Solutions to EE106 Problem set 5

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P.1

If we do go ahead and use the product rule, i.e.

$$\frac{d}{dx}\left(e^{(\cos(x))^2}e^{(\sin(x))^2}\right) = \left(\frac{d}{dx}e^{(\cos(x))^2}\right)e^{(\sin(x))^2} + e^{(\cos(x))^2}\left(\frac{d}{dx}e^{(\sin(x))^2}\right)$$

What are these two derivatives? Two applications of the chain rule will give them. First, if $f(x) = e^x$, $g(x) = x^2$ and $h(x) = \cos(x)$, then we see that

$$e^{(\cos(x))^2} = f(g(h(x)))$$

so that

$$\frac{d}{dx}e^{(\cos(x))^2} = f'(g(h(x)))\frac{d}{dx}g(h(x))$$

$$= f'(g(h(x)))g'(h(x))h'(x)$$

$$= f'((\cos(x))^2)\, g'(\cos(x))\, h'(x)$$

$f'(x) = e^x$, $g'(x) = 2x$ and $h'(x) = -\sin(x)$, so

$$\frac{d}{dx}e^{(\cos(x))^2} = -2e^{(\cos(x))^2}\sin(x)\cos(x)$$

A very similar calculation gives

$$\frac{d}{dx}e^{(\sin(x))^2} = 2e^{(\sin(x))^2}\sin(x)\cos(x)$$

so that

$$\frac{d}{dx}\left(e^{(\cos(x))^2}e^{(\sin(x))^2}\right) = \left(-2e^{(\cos(x))^2}\sin(x)\cos(x)\right)e^{(\sin(x))^2} + e^{(\cos(x))^2}\left(2e^{(\sin(x))^2}\sin(x)\cos(x)\right)$$

$$\boxed{= 0}.$$

But this makes perfect sense: recall that $e^Ae^B = e^{A+B}$ for all A and B, and that $(\sin(x))^2+(\cos(x))^2 = 1$; thus,

$$e^{(\cos(x))^2}e^{(\sin(x))^2} = e^{(\sin(x))^2+(\cos(x))^2} = e^1 = e = 2.7182818\ldots$$

which is a <u>constant</u>. And the derivative of any constant is zero, so

$$\frac{d}{dx}\left(e^{(\cos(x))^2}e^{(\sin(x))^2}\right) = \frac{d}{dx}(e)\ \boxed{= 0}.$$

Thus by using the properties of the exponential function and the most basic trigonometry identity, there was no need to use the product rule (although you certainly can if you like).

P2.

There's a hard way to do this and an easy way to do this. The hard way is simply to use Taylor's formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f(x) = \frac{1}{1+x^2}$. The first few derivatives are

$$f^{(0)}(x) = f(x) = \frac{1}{1+x^2} \Rightarrow f^{(0)}(0) = 1$$

$$f^{(1)}(x) = f'(x) = -\frac{2x}{(1+x^2)^2} \Rightarrow f^{(1)}(0) = 0$$

$$f^{(2)}(x) = f''(x) = \frac{6x^2-2}{(1+x^2)^3} \Rightarrow f^{(2)}(0) = -2$$

$$f^{(3)}(x) = \frac{d^3f}{dx^3}(x) = -\frac{24(x^3-x)}{(1+x^2)^4} \Rightarrow f^{(3)}(0) = 0$$

$$f^{(4)}(x) = \frac{d^4f}{dx^4}(x) = \frac{24(5x^4-10x^2+1)}{(1+x^2)^5} \Rightarrow f^{(4)}(0) = 24$$

and so on. So we needed the first <u>five</u> terms in the series, to get the first three <u>nonzero</u> terms, which are

$$\frac{1}{1+x^2} = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$= \frac{1}{1}\cdot 1 + \frac{0}{1}x + \frac{(-2)}{2}x^2 + \frac{0}{6}x^3 + \frac{(24)}{24}x^4 + \ldots$$

$$\boxed{= 1 - x^2 + x^4 + \ldots}$$

The easy way is using something we already know: if $f(x) = \frac{1}{1-x}$, then it has the Taylor expansion

$$f(x) = 1 + x + x^2 + x^3 + \ldots$$

The function we have here is $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = f(-x^2)$ so putting $(-x^2)$ in for (x) in the above expansion gives

$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \ldots$$

$$\boxed{= 1 - x^2 + x^4 - x^6 + \ldots}$$

which again gives the result we want.

P.3

The first three terms of the given expansion are

$$\ln(x) \approx \frac{(-1)^{1+1}(x-1)^1}{1} + \frac{(-1)^{2+1}(x-1)^2}{2} + \frac{(-1)^{3+1}(x-1)^3}{3}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

So if $x = e^{-1} \approx 0.36788$, then

$$\ln(e^{-1}) \approx (0.36788-1) - \frac{1}{2}(0.36788-1)^2 + \frac{1}{3}(0.36788-1)^3$$

$$= -0.63212 - \frac{1}{2}(-0.63212)^2 + \frac{1}{3}(-0.63212)^3$$

$$\approx -0.91610$$

This is rather close to one; more precisely, it's within

$$\frac{-1+1(-0.91610)}{-1} = 0.083899 \approx \boxed{8.39\%}$$

of the correct answer. Not too bad, but perhaps not as exact as we'd like. You can convince yourself that using more terms in the expansion will give a better result.

P.4

This is all about manipulating complex numbers, specifically, the complex number $z = 2+3i$.

First, z^*: recall that taking a complex conjugate simply flips the signs of all is, namely, $i \leftrightarrow -i$. Thus,

$$\boxed{z^* = 2-3i}$$

Now, z^2: we just multiply as usual, remembering that $i^2 = -1$, so that

$$z^2 = (2+3i)^2 = (2)^2 + 2\cdot(2)(3i) + (3i)^2$$
$$= 4 + 12i + 9i^2 = 4 + 12i - 9$$
$$= \boxed{-5+12i}$$

e^z: since $z = 2+3i$, then $e^z = e^{2+3i} = e^2 e^{3i}$. The first is just $(2.71828\ldots)^2$; for the second, we need to use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, so

$$e^{3i} = \cos(3) + i\sin(3).$$

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Remember that we must use <u>radians</u> and not <u>degrees</u> when finding sines and cosines, so using a calculator to put the numbers in, we find

$$e^{2+3i} = e^2(\cos(3) + i\sin(3))$$
$$= (7.3890561)(-0.98999 + 0.14112i)$$
$$= \boxed{-7.315 + 1.043i}$$