Solutions to EE 106 Problem Set 4

1

P.1

This is done in exactly the same way that we find $\frac{d}{dx}\arcsin(x)$, namely, by differentiating both sides of the eq'n
$\tan(\arctan(x)) = x$.

If $f(x) = \arctan(x)$ and $g(x) = \tan(x)$, then the left-hand side has the form $g(f(x))$. The chain rule thus says
$\frac{d}{dx}\tan(\arctan(x)) = g'(f(x))\, f'(x)$

The derivative of $\tan(x)$ is $(\sec(x))^2$, so

$$\frac{d}{dx}\tan(\arctan(x)) = (\sec(f(x)))^2 \frac{d}{dx}\arctan(x)$$
$$= (\sec(\arctan(x)))^2 \frac{d}{dx}\arctan(x).$$

But there's an identity relating the secant to the tangent:
$$(\sin(x))^2 + (\cos(x))^2 = 1 \Rightarrow \frac{(\sin(x))^2 + (\cos(x))^2}{(\cos(x))^2} = \frac{1}{(\cos(x))^2}$$
or
$$(\tan(x))^2 + 1 = (\sec(x))^2.$$
Thus,
$$(\sec(\arctan(x)))^2 = 1 + (\tan(\arctan(x)))^2$$
But $\tan(\arctan(x))$ is just x, so $(\sec(\arctan(x)))^2 = 1 + x^2$.
Thus,
$$\frac{d}{dx}\left(\tan(\arctan(x))\right) = (1+x^2)\frac{d}{dx}\arctan(x).$$
But $\tan(\arctan(x)) = x$, so its derivative is 1, giving
$$1 = (1+x^2)\frac{d}{dx}[\arctan(x)]$$
or
$$\boxed{\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}}$$
as desired.

P.2

The key here is to first compute $f'(x)$, then to see where it's equal to zero. Then we compute $f''(x)$; if $f''(x) > 0$ at one of the critical points, it's a minimum. $f''(x) < 0$ means a maximum and $f''(x) = 0$ gives no information. But to find the inflection points, we look to see where $f''(x)$ is zero and if it changes sign across such points. If so, it's a point of inflection. So let's go:

(a) $f(x)=x^3-18x^2-39x+100$, so $f'(x)=3x^2-36x-39$ and $f''(x)=6x-36$.

What points a satisfy $f(a)=0$? These are the sol'ns to the quadratic eq'n $3a^2-36a-39=0$, i.e. the two points a_+ and a_- given by

$$a_+ = \frac{36+\sqrt{36^2-4\cdot3\cdot(-39)}}{2\cdot3} = \frac{36+\sqrt{1296+468}}{6} = \frac{36+\sqrt{1764}}{6} = \frac{36+42}{6} = 13$$

$$a_- = \frac{36-42}{6} = -1$$

Now, we evaluate $f''(x)$ at each of these:

$f''(13) = 6\cdot13-36 = 78-36 = 42$

$f''(-1) = 6\cdot(-1)-36 = -42$

So the critical points are $x=13$ (a minimum) and $x=-1$ (a maximum).

Any points of inflection must satisfy $f''(x)=6x-36=0$, and we see only $x=6$ works. But notice that for $x<6$, $f''(x)<0$ and for $x>6$, $f''(x)>0$, so $f''(x)$ does indeed change sign at $x=6$, so there is a point of inflection there.

(b) $f(x)=\frac{x}{x^2+1}$, $f'(x)=\frac{1\cdot(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$, $f''(x)=\frac{(-2x)(x^2+1)^2-(-x^2+1)2(2x)(x^2+1)}{(x^2+1)^4}$

$$= \frac{2x^3-6x}{(x^2+1)^3}.$$

The critical points ~~are~~ where $f'(x)$ vanishes, i.e. points a for which $\frac{-a^2+1}{(a^2+1)^2}=0$. $\frac{1}{(a^2+1)^2}$ is never zero, so this requires $-a^2+1=0$, or $a^2=1$, namely, $a_+=1$, $a_-=-1$. Putting these into $f''(x)$ gives

$f''(1) = \frac{2\cdot1-6\cdot1}{(1+1)^3} = -\frac{1}{2}$ and $f''(-1) = \frac{2(-1)-6(-1)}{(1+1)^2} = \frac{1}{2}$, so

$x=1$ is a maximum and $x=-1$ is a minimum.

Now we see where $f''(x)$ vanishes: $\frac{2x^3-6x}{(x^2+1)^3}=0$. $\frac{1}{(x^2+1)^3}$ is never zero, so $2x^3-6x=2x(x^2-3)$ must vanish. This happens if either $x=0$ or $x^2=3$, i.e. three points $x=0$, $x=\sqrt{3}$ and $x=-\sqrt{3}$. Does $f''(x)$ change sign across any of these?

$(x^2+1)^3$ is always positive, so $f''(x)$ changes sign when $x(x^2-3)$ changes sign. Near zero, x^2-3 is negative. So if x is slightly less than zero, $x(x^2-3)>0$, and if x is slightly greater than zero, $x(x^2-3)<0$. Thus, $f''(x)$ changes sign at zero, so $\boxed{x=0 \text{ is a point of inflection}}$

Near $x=\sqrt{3}$, $x(x+\sqrt{3})$ is positive, so the sign of $x(x^2-3)$ is the same as that of $x-\sqrt{3}$. Since this is negative if $x<\sqrt{3}$ and positive if $x>\sqrt{3}$, $f''(x)$ changes sign at $\boxed{x=\sqrt{3}}$, $\boxed{\text{so it's also an inflection point.}}$

A similar argument goes for $x=-\sqrt{3}$: $x(x-\sqrt{3})$ is positive near $-\sqrt{3}$ (since both x and $x-\sqrt{3}$ are negative), so the sign of $x(x^2-3)$ is the same as that of $x+\sqrt{3}$. If $x<-\sqrt{3}$, this is negative and if $x>-\sqrt{3}$, it's positive, so $\boxed{x=-\sqrt{3} \text{ is a point of inflection.}}$

(c) $f(x)=x^6$, $f'(x)=6x^5$ and $f''(x)=30x^4$. $f'(a)=6a^5=0$ only if $a=0$, and $f''(0)=0$ means that even though $\boxed{0 \text{ is a critical point, we don't know if it's a maximum or minimum}}$. (At least, not from using $f''(x)$; if you sketch it, you'll see it's a minimum.)

$f''(x)=30x^4$ vanishes only at $x=0$, but $x^4>0$ if x is either positive or negative. Thus, $f''(x)$ does not change sign at zero, so $\boxed{\text{There are no points of inflection.}}$

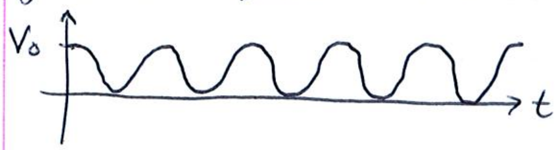
(d) $f(x)=e^{-x^2}$, $f'(x)=-2xe^{-x^2}$, $f''(x)=4x^2e^{-x^2}-2e^{-x^2}$. e^{-x^2} is always a positive number, so $f'(x)$ can only be zero if $x=0$. And since $f''(0)=-2$, we conclude that $\boxed{\text{There is a maximum at } x=0.}$

Since we can write $f''(x)=2(2x^2-1)e^{-x^2}$, then this can vanish only if $2x^2-1=0$, or $x=1/\sqrt{2}$ or $x=-\frac{1}{\sqrt{2}}$. $2e^{-x^2}>0$, so the sign of $f''(x)$ ~~dep~~ is the same as that of $2x^2-1=2(x+1/\sqrt{2})(x-1/\sqrt{2})$. Near $x=\frac{1}{\sqrt{2}}$, $2(x+\frac{1}{\sqrt{2}})$ is positive, and since $x-\frac{1}{\sqrt{2}}<0$ for $x<1/\sqrt{2}$ and

$x - 1/\sqrt{2} > 0$ for $x > 1/\sqrt{2}$, $f''(x)$ changes sign so $\boxed{x = 1/\sqrt{2} \text{ is an inflection point.}}$
And an exactly similar argument will also quickly tell us that there is $\boxed{\text{a point of inflection at } x = -1/\sqrt{2}}$ as well.

P. 3

The key here is not just to find the critical points of $V(t)$ and see if they're maxima or minima, but also to find the value of $V(t)$ itself at those points. A quick sketch of $V(t)$ gives us a good idea of what we expect:



So we expect V_0 to be the maximum voltage and 0 to be the minimum voltage. How do we do this mathematically? Well, we find the points where $\frac{dV}{dt} = 0$, use $\frac{d^2V}{dt^2}$ to determine if they're maxima or minima, then put these values back into V to get the max and min voltages.

We could do this by taking $\frac{d}{dt}\left[V_0(\cos(\omega t))^2\right]$, but it's a good idea to get all those trigonometric identities memorized, so let's do it an easier way: recall that one of these trig identities is, for any angle θ,

$$(\cos(\theta))^2 = \frac{1}{2}\left[1 + \cos(2\theta)\right]$$

(This is a very handy identity, so try to remember it.) This means we can rewrite $V(t)$ as

$$V(t) = V_0\left[\frac{1}{2}\left(1 + \cos(2\omega t)\right)\right] = \frac{V_0}{2} + \frac{V_0}{2}\cos(2\omega t).$$

Now, taking the derivative is easier than it would have been in its original form, because the constant part $V_0/2$ differentiates to zero, i.e.

$$\frac{dV}{dt}(t) = \frac{d}{dt}\left[\frac{V_0}{2} + \frac{V_0}{2}\cos(2\omega t)\right] = -V_0\omega\sin(2\omega t)$$

So this vanishes if $\sin(2\omega t)$ vanishes. $\sin x = 0$ whenever x is a multiple of π, so the values of t which give $\frac{dV}{dt} = 0$ are

$$t = 0, \pm\frac{\pi}{2\omega}, \pm\frac{2\pi}{2\omega}, \pm\frac{3\pi}{2\omega}, \pm\frac{4\pi}{2\omega}, \text{ etc.}$$

Now, $\frac{d^2V}{dt^2} = -2V_0\omega^2\cos(2\omega t)$. If $2\omega t$ is a multiple of π, then $\cos(2\omega t)$ is either $+1$ or -1. In particular, an even multiple of π gives $\cos(2\omega t) = 1$ and an odd multiple gives $\cos(2\omega t) = -1$.

Here, $\frac{d^2V}{dt^2} > 0$ when

$t = \pm \frac{\pi}{2\omega}, \pm \frac{3\pi}{2\omega}, \pm \frac{5\pi}{2\omega}, \ldots$ so these are minima.

If $\frac{d^2V}{dt^2} < 0$, the t must be one of

$t = 0, \pm \frac{2\pi}{2\omega}, \pm \frac{4\pi}{2\omega}, \pm \frac{6\pi}{2\omega}, \ldots$ and these are maxima.

At the minima, $\frac{d^2V}{dt^2} > 0 \Rightarrow \cos(2\omega t) = -1$. Thus, for any of these times, the value of V is

$$V = \frac{V_0}{2} + \frac{V_0}{2}\cos(2\omega t) = \frac{V_0}{2} - \frac{V_0}{2} \boxed{= 0}$$

as we expected from the plot. $\frac{d^2V}{dt^2} < 0$ requires $\cos(2\omega t) = 1$ at the critical points, so these maxima are

$$V = \frac{V_0}{2} + \frac{V_0}{2}\cos(2\omega t) = \frac{V_0}{2} + \frac{V_0}{2} \boxed{= V_0}$$

again, as we expected.

P.4

Now, (a) and (b) can be figured out without l'Hôpital's rule: Since $5x^3+4x-7$ is very close to $5x^3$ when x is large (since $x^3 >> x >> 1$) and $7x^3+5x^2-x+1 \approx 7x^3$ (for the same reason), then $\frac{5x^3+4x-7}{7x^3+5x^2-x+1} \approx \frac{5x^3}{7x^3} = 5/7$. Thus, expect the limit for (a) to be 5/7. Similarly, for small x, the $x/7$ and $x^2/2$ terms are very small compared respectively to the $11/x^2$ and $-2/x^2$, so we expect the ratio to be very close to $\frac{11/x^2}{-2/x^2} = -11/2$, and thus this is the limit.

However, it's still useful to do these with l'Hôpital's rule, because it will work even if you don't see the limits as I just described them. Plus, for (c), it's very hard to see how to get the limit any other way. So let's do it:

(a) $\lim_{x\to\infty} \frac{5x^3+4x-7}{7x^3+5x^2-x+1}$: both numerator and denominator diverge, so we can indeed use l'Hôpital's rule. So a first application says

$$\lim_{x\to\infty} \frac{5x^3+4x-7}{7x^3+5x^2-x+1} = \lim_{x\to\infty} \frac{15x^2+4}{21x^2+10x-1}.$$

But this is still an indeterminate limit, so apply the rule again:

$$\lim_{x\to\infty} \frac{15x^2+4}{21x^2+10x-1} = \lim_{x\to\infty} \frac{30x}{42x+10}.$$

This still looks like $\frac{\infty}{\infty}$, so one more time:

$$\lim_{x\to\infty}\frac{30x}{42x+10} = \lim_{x\to\infty}\frac{30}{42} = \frac{30}{42} = \frac{5}{7}.$$

Thus, we see

$$\boxed{\lim_{x\to\infty}\frac{5x^3+4x-7}{7x^3+5x^2-x+1} = 5/7.}$$

(b) As $x\to 0$, this also has an $\frac{\infty}{\infty}$ form, so we can use l'Hôpital's rule:

$$\lim_{x\to 0}\frac{x/7+11/x^2}{x^2/2-2/x^2} = \lim_{x\to 0}\frac{1/7-22/x^3}{x+4/x^3}.$$

still $\frac{\infty}{\infty}$, so again:

$$\lim_{x\to 0}\frac{1/7-22/x^3}{x+4/x^3} = \lim_{x\to 0}\frac{66/x^4}{1+12/x^4} = \lim_{x\to 0}\frac{66}{x^4+12}$$

and now the limits of both top and bottom are finite, so $-66/12$ is the limit. Thus,

$$\boxed{\lim_{x\to 0}\frac{x/7+11/x^2}{x^2/2-2/x^2} = -\frac{11}{2}.}$$

(c) Use the hint given: let $y=1/x$, then $y\to 0$ as $x\to\infty$. Thus, replacing x by $1/y$ throughout gives

$$\lim_{x\to\infty}\left[(x+27)^{1/3}-x^{1/3}\right] = \lim_{y\to 0}\left[\left(\frac{1}{y}+27\right)^{1/3}-\left(\frac{1}{y}\right)^{1/3}\right]$$

Now,

$$\left(\frac{1}{y}+27\right)^{1/3} = \left(\frac{1+27y}{y}\right)^{1/3} = \frac{(1+27y)^{1/3}}{y^{1/3}}$$

so we can rewrite this as

$$\lim_{x\to\infty}\left[(x+27)^{1/3}-x^{1/3}\right] = \lim_{y\to 0}\left[\frac{(1+27y)^{1/3}-1}{y^{1/3}}\right]$$

As $y\to 0$, the fraction has a $0/0$ form, so we can indeed use l'Hôpital's rule:

$$\lim_{y\to 0}\left[\frac{(1+27y)^{1/3}-1}{y^{1/3}}\right] = \lim_{y\to 0}\frac{\frac{d}{dy}\left[(1+27y)^{1/3}-1\right]}{\frac{d}{dy}y^{1/3}} = \lim_{y\to 0}\frac{27\cdot\frac{1}{3}(1+27y)^{1/3-1}}{\frac{1}{3}y^{1/3-1}}$$

$$= \lim_{y\to 0} 27\frac{(1+27y)^{-2/3}}{y^{-2/3}} = \lim_{y\to 0}\frac{27y^{2/3}}{(1+27y)^{2/3}}.$$

Now the top goes to 0 but the bottom goes to 1, so the overall limit is 0. Therefore,

$$\boxed{\lim_{x\to\infty}\left[(x+27)^{1/3}-x^{1/3}\right] = 0.}$$