Solutions to EE106 Problem Set 3

P.1

These functions are all combinations of terms of the form x^n, and since $\frac{d}{dx}x^n = nx^{n-1}$, they're all easily doable:

(a) $\frac{d}{dx}(2x^{10} - x^7 + 3/x^{25}) = \frac{d}{dx}(2x^{10} - x^7 + 3x^{-25})$

$$= 2(10x^{10-1}) - (7x^{7-1}) + 3(-25x^{-25-1})$$

$$= \boxed{20x^9 - 7x^6 - \frac{75}{x^{26}}}$$

(b) We could multiply the two functions explicitly and then do the derivative, but let's use the product rule instead (the answer will be the same either way):

$$\frac{d}{dx}\left[(x^4 + 22x^5 - 102x^2)\left(x - \frac{23}{x}\right)\right] = \left[\frac{d}{dx}(x^4 + 22x^5 - 102x^2)\right]\left(x - \frac{23}{x}\right) + (x^4 + 22x^5 - 102x^2)\frac{d}{dx}(x - 23x^{-1})$$

$$= \left[(4x^3) + 22(5x^4) - 102(2x^1)\right]\left(x - \frac{23}{x}\right) + (x^4 + 22x^5 - 102x^2)\left[(1 \cdot x^0) - 23(-1x^{-2})\right]$$

$$= \boxed{(4x^3 + 110x^4 - 204x)\left(x - \frac{23}{x}\right) + (x^4 + 22x^5 - 102x^2)\left(1 + \frac{23}{x^2}\right)}$$

or, if we multiply everything out and collect terms,

$$\frac{d}{dx}\left[(x^4 + 22x^5 - 102x^2)\left(x - \frac{23}{x}\right)\right] = \boxed{132x^5 + 5x^4 - 2024x^3 - 375x^2 + 2692}$$

P.2

For these, we use the chain rule, although the second one can be done in a much easier way as well...

(a) First, let $f(x) = e^x$ and $g(x) = e^{\cos(x)}$. The function given thus has the form $f(g(x))$. Thus,

$$\frac{d}{dx}\left[e^{e^{\cos(x)}}\right] = f'(g(x))\,g'(x) = e^{e^{\cos(x)}}\frac{d}{dx}\left[e^{\cos(x)}\right].$$

To evaluate the second term, we need the chain rule again: let $h(x) = e^x$ and $k(x) = \cos(x)$, so $e^{\cos(x)} = h(k(x))$. Thus,

$$\frac{d}{dx}e^{\cos(x)} = h'(k(x))\,k'(x) = e^{\cos(x)}(-\sin(x))$$
$$= -e^{\cos(x)}\sin(x).$$

Therefore, we're done:

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$$\frac{d}{dx}\left[e^{e^{\cos(x)}}\right] = e^{e^{\cos(x)}} \cdot \left(-e^{\cos(x)}\sin(x)\right)$$

$$= \boxed{-\exp\left[e^{\cos(x)} + \cos(x)\right]\sin(x)}$$

(b) First, the longer way, using the chain rule: if $f(x) = 3e^x$ and $g(x) = \ln(1/x)$, then

$$\frac{d}{dx}\left[3e^{\ln(1/x)}\right] = f'(g(x))\,g'(x) = 3e^{\ln(1/x)} \frac{d}{dx}\left(\ln\left(\frac{1}{x}\right)\right).$$

And we use the chain rule again, with $h(x) = \ln|x|$ and $k(x) = \frac{1}{x}$:

$$\frac{d}{dx}\left(\ln(1/x)\right) = h'(\cancel{x}k(x))\,k'(x)$$

$$= \frac{1}{(1/x)} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x}$$

so

$$\boxed{\frac{d}{dx}\left[3e^{\ln(1/x)}\right] = \frac{-3e^{\ln(1/x)}}{x}}$$

However, recall that exp and ln are inverse functions to one another, so $e^{\ln(1/x)} = \frac{1}{x}$. Thus our original function was $\frac{3}{x}$, and the derivative of this is $-3/x^2$. And this agrees with what's in the box above: $-3e^{\ln(1/x)}/x = -3(1/x)/x = \boxed{-3/x^2}$.

Either way is doable.

P.3

Much like $\sin(x)$, $\cos(x)$, $\tan(x)$, etc. all have various identities associated with them (e.g. $(\sin(x))^2 + (\cos(x))^2 = 1$), so do the hyperbolic functions. However, these functions are all defined in terms of exponentials, so their identities can be easy to prove, as we see here:

(a) Recall that $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

Thus,

$$(\cosh(x))^2 = \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2 = \left(\frac{1}{2}e^x\right)^2 + 2\left(\frac{1}{2}e^x\right)\left(\frac{1}{2}e^{-x}\right) + \left(\frac{1}{2}e^{-x}\right)^2$$

$$= \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}$$

and

$$(\sinh(x))^2 = \left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)^2 = \left(\frac{1}{2}e^x\right) + 2\left(\frac{1}{2}e^x\right)\left(-\frac{1}{2}e^{-x}\right) + \left(-\frac{1}{2}e^{-x}\right)^2$$

$$= \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}.$$

Thus, if we subtract the second from the first, we get

$$(\cosh(x))^2 - (\sinh(x))^2 = \left(\tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}\right) - \left(\tfrac{1}{4}e^{2x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}\right)$$
$$= \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) \boxed{= 1}$$

as desired.

(b) We compute the left-hand side first:

$$2\sinh(x)\cosh(x) = 2\left(\tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x}\right)\left(\tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}\right)$$
$$= 2\left(\tfrac{1}{4}e^{2x} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4}e^{-2x}\right)$$
$$= \tfrac{1}{2}e^{2x} - \tfrac{1}{2}e^{-2x}.$$

But $\sinh(2x) = \tfrac{1}{2}e^{(2x)} - \tfrac{1}{2}e^{-(2x)} = \tfrac{1}{2}e^{2x} - \tfrac{1}{2}e^{-2x}$, exactly as we see above. Thus,

$$\boxed{2\sinh(x)\cosh(x) = \sinh(2x)}$$

P.4

This proof requires <u>two</u> applications of the product rule, as follows: this rule states that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

so

$$\frac{d^2}{dx^2}[f(x)g(x)] = \frac{d}{dx}\left\{\frac{d}{dx}[f(x)g(x)]\right\}$$
$$= \frac{d}{dx}[f'(x)g(x) + f(x)g'(x)]$$
$$= \frac{d}{dx}[f'(x)g(x)] + \frac{d}{dx}[f(x)g'(x)]$$

Using the product rule on each term gives

$$\frac{d^2}{dx^2}[f(x)g(x)] = \left[\left(\frac{d}{dx}f'(x)\right)g(x) + f'(x)\left(\frac{d}{dx}g(x)\right)\right] + \left[\left(\frac{d}{dx}f(x)\right)g'(x) + f(x)\left(\frac{d}{dx}g'(x)\right)\right]$$
$$= f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)$$
$$= \boxed{f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)}$$

as desired.