Solutions to EE106 Problem Set 2

P.1

What we're looking for is a series of the form

$$7 + 7r + 7r^2 + 7r^3 + \cdots$$

such that its total sum is 23. This is actually quite straightforward: recall that $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, where a is the first term in the series and r the ratio between successive terms. We have a already; it's 7, as stated in the problem. And since the total sum is 23, we just need an r such that

$$\frac{7}{1-r} = 23$$

so

$$1 - r = \frac{7}{23} \Rightarrow r = 1 - \frac{7}{23} = \frac{16}{23}$$

so $\boxed{a = 7}$ and $\boxed{r = 16/23}$ give the infinite geometric series

$$7 + 7\cdot\frac{16}{23} + 7\cdot\left(\frac{16}{23}\right)^2 + 7\cdot\left(\frac{16}{23}\right)^3 + \cdots = 23$$

as desired.

P.2

(a) The form of the series suggests that we might compare it to $\sum_{n=1}^{\infty} \frac{1}{3^n}$, and that's exactly what we can do: for $n = 1, 2, 3, \ldots$, $2n$ is a positive number, so

$$3^n + 2n \geq 3^n$$

which means $\frac{1}{3^n+2n}$ is smaller than $\frac{1}{3^n}$. And since $\frac{1}{3^n+2n}$ is positive, we have

$$0 \leq \frac{1}{3^n+2n} \leq \frac{1}{3^n}$$

Now, $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$

is an infinite geometric series with $a = \frac{1}{3}$ and $r = 1/3$; since $|r| < 1$, it converges to

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1/3}{1 - 1/3} = \frac{1}{2}$$

So we have everything we need for the comparison test: if $a_n = \frac{1}{3^n+2n}$ and $b_n = \frac{1}{3^n}$, then $0 \leq a_n \leq b_n$, and since $\sum_{n=1}^{\infty} b_n$ converges, we conclude

$\boxed{\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3^n+2n} \text{ also converges.}}$

(b) Let $s_n = \frac{(-1)^n x^{2n}}{(2n)!}$, so the series we want to check is $\sum_{n=0}^{\infty} s_n$. For the ratio test, we need to find $|s_{n+1}/s_n|$ and see how it behaves as $n \to \infty$; if its limit is less than 1, then it will converge.

We see

$$s_{n+1}/s_n = \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}} = \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{(-1)^n x^{2n}}{(2n)!}} = \frac{(-1)^{n+1}}{(-1)^n} \frac{x^{2n+2}}{x^{2n}} \frac{(2n)!}{(2n+2)!}$$

The first two factors are -1 and x^2 respectively. The third we can find by using the def'n of the factorial:

$$\frac{(2n)!}{(2n+2)!} = \frac{(2n)(2n-1)(2n-2)\ldots 2\cdot 1}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\ldots 2\cdot 1} = \frac{1}{(2n+2)(2n+1)}$$

since all other terms cancel out. Thus,

$$\left|\frac{s_{n+1}}{s_n}\right| = \left|\frac{-x^2}{(2n+2)(2n+1)}\right| = \frac{x^2}{(2n+2)(2n+1)}.$$

The numerator is a fixed value, but as $n \to \infty$, the denominator grows without bound. Thus,

$$r = \lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = 0$$

and since 0 is less than 1, $\boxed{\text{this series converges.}}$

(In fact, it's a very familiar series: it's equal to $\cos(x)$, as we'll see shortly in lecture.)

P.3

The first two are straightforward:

(a) $\lim_{x\to 1} \frac{x^2-25}{x+5} = \frac{1-25}{1+5} = -\frac{24}{6} = \boxed{-4}$

(b) $\lim_{x\to 2} \frac{x^3-343}{x-7} = \frac{8-343}{2-7} = \frac{-335}{-5} = \boxed{67}$

(c) This looks bad: both the numerator and denominator go to 0. Now, we just learned l'Hôpital's rule, so we know how to handle limits like this, but we didn't know it when this problem was assigned. However, we don't need it if we realise that

$$\frac{1-x}{1-\sqrt{x}} = \frac{1-(\sqrt{x})^2}{1-(\sqrt{x})} = \frac{(1+(\sqrt{x}))(1-(\sqrt{x}))}{1-(\sqrt{x})} = 1+\sqrt{x}$$

and so

$$\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} = 1+1 = \boxed{2}$$

Actually, we could have used this exact same trick – factorization – to do the first two as well:

$$\frac{x^2-25}{x+5} = \frac{(x-5)(x+5)}{x+5} = x-5 \Rightarrow \lim_{x\to 1} \frac{x^2-25}{x+5} = 1-5 = -4$$

$$\frac{x^3-343}{x-7} = \frac{(x-7)(x^2+7x+49)}{x-7} = x^2+7x+49 \Rightarrow \lim_{x\to 2} \frac{x^3-343}{x-7} = 4+14+49=67$$

But now we can also do (c) using l'Hôpital's rule:

$$\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\frac{d}{dx}(1-x)}{\frac{d}{dx}(1-x^{1/2})} = \lim_{x\to 1} \frac{-1}{-\frac{1}{2}x^{-1/2}} = \lim_{x\to 1} 2\sqrt{x} = 2.$$

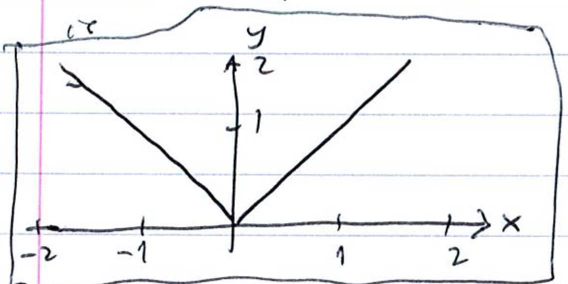
So many limits have multiple ways in which you can evaluate them.

P.4

(a) $f(x)=|x|$ can be written as a piecewise function:

$$f(x)=|x| = \begin{cases} -x & x<0 \\ x & x\geq 0 \end{cases}$$

since $-x$ is positive if x is negative. To the left of $x=0$, it's a line of slope -1 going into the origin; to the right, it's a line of slope 1 going from the origin. Thus, its plot is



Now, for $f(x)$ to be continuous at zero, then its limit coming from either side of $x=0$ must be equal to $f(0)$. $f(0)=0$, of course, and we see that

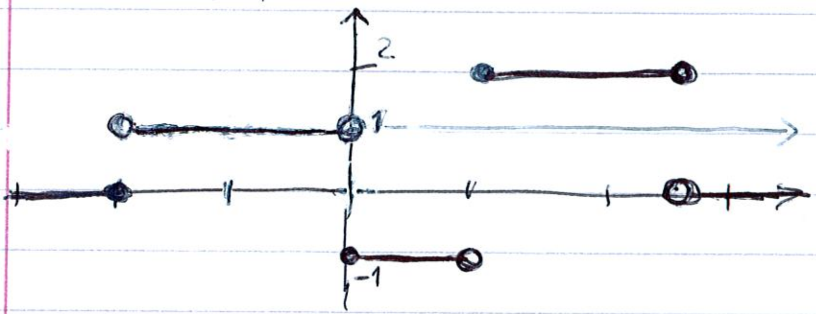
$$\lim_{x\to 0^-}(-x) = 0 \quad \text{and} \quad \lim_{x\to 0^+}(x) = 0$$

so $\boxed{\text{it is continuous at } x=0}$. We didn't have to lift our pen from the paper in drawing this. So a sharp point in a plot does not mean it's discontinuous there; as long as there's no jump in the value of $f(x)$, it's continuous.

(b) For this part, we do want a function that jumps at $-2, 0, 1$ and 2.5. Any function that requires us to lift our pen off the paper will do; the one I've chosen is

$$f(x) = \begin{cases} 0 & x \leq -2 \\ 1 & -2 < x < 0 \\ -1 & 0 \leq x < 1 \\ 2 & 1 \leq x \leq 2.5 \\ 0 & x > 2.5 \end{cases}$$

which looks like



(I've used open and closed circles to make it clear exactly what $f(-2)$, $f(0)$, $f(1)$ and $f(2.5)$ are.) You were, of course, free to choose any function, as long as it has jumps at the four values of x specified.