Solutions to EE106 Problem Sheet 10

P.1

The mean value of a function $f(x)$ over an interval $[a,b]$ is given by

$$\langle f \rangle = \frac{\int_a^b f(x)\,dx}{b-a}$$

and represents an "average" value of the function between a and b.

This problem has $f(x) = x^5 - 2/x^3$ whose integral is

$$\int_a^b f(x)\,dx = \int_1^2 \left(x^5 - \frac{2}{x^3}\right)dx = \int_1^2 x^5\,dx - 2\int_1^2 x^{-3}\,dx$$

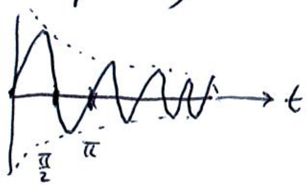
$$= \left[\frac{1}{5+1}x^{5+1}\right]_1^2 - 2\left[\frac{1}{-3+1}x^{-3+1}\right]_1^2$$

$$= \frac{1}{6}\left[(2)^6 - 1^6\right] - 2\cdot\left(-\frac{1}{2}\right)\left[2^{-2} - 1^{-2}\right]$$

$$= \frac{63}{6} + \left(\frac{1}{4} - 1\right) = \boxed{39/4}$$

P.2

The voltage given is truly a good way of describing an exponentially decaying AC voltage: as a function of time it looks like:



We want to find the RMS value of $V(t) = V_0 e^{-t}\sin(2t)$;

i.e. $V_{RMS} = \sqrt{\langle V^2 \rangle}$. First we find $\langle V^2 \rangle$:

$$\langle V^2 \rangle = \frac{\int_0^\pi (V(t))^2\,dt}{\pi - 0} = \frac{1}{\pi}\int_0^\pi V_0^2 e^{-2t}(\sin(2t))^2\,dt$$

$$= \frac{V_0^2}{2\pi}\int_0^\pi e^{-2t}(1 - \cos(4t))\,dt$$

where we used $\sin^2(2t) = \frac{1}{2} - \frac{1}{2}\cos(4t)$. There are two integrals we must do here:

$$\int_0^\pi e^{-2t}\,dt = \left[-\frac{1}{2}e^{-2t}\right]_0^\pi = -\frac{1}{2}\left(e^{-2\pi} - e^0\right) = \frac{1}{2}\left(1 - e^{-2\pi}\right)$$

(which, since $e^{-2\pi} < 1$, is positive). The other, $\int_0^\pi e^{-2t}\cos(4t)\,dt$, requires two integration by parts and some algebra: letting $f(t) = e^{-2t}$ and $g'(t) = \cos(4t)$, we get

$$\int_0^\pi e^{-2t}\cos(4t)\,dt = \left[f(t)g(t)\right]_0^\pi - \int_0^\pi f'(t)g(t)\,dt.$$

$f'(t) = -2e^{-2t}$ and $g(t) = \frac{1}{4}\sin(4t)$ are easy to see, so

$$\int_0^\pi e^{-2t}\cos(4t)\,dt = \left[e^{-2t}\cdot\tfrac{1}{4}\sin(4t)\right]_0^\pi - \int_0^\pi(-2e^{-2t})\left(\tfrac{1}{4}\sin(4t)\right)dt$$

$$= \frac{1}{2}\int_0^\pi e^{-2t}\sin(4t)\,dt$$

since $\sin(4\pi) = \sin(0) = 0$. Another integration by parts, this time with $f(t) = e^{-2t}$ and $g'(t) = \sin(4t)$ (and thus $f'(t) = -2e^{-2t}$ and $g(t) = -\frac{1}{4}\cos(4t)$) gives

$$\int_0^\pi e^{-2t}\cos(4t)\,dt = \frac{1}{2}\left\{\left[e^{-2t}\cdot\left(-\tfrac{1}{4}\cos(4t)\right)\right]_0^\pi - \int_0^\pi(-2e^{-2t})\left(-\tfrac{1}{4}\cos(4t)\right)dt\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{4}e^{-2\pi}\cos(4\pi) + \frac{1}{4}e^{0}\cos(0) - \frac{1}{2}\int_0^\pi e^{-2t}\cos(4t)\,dt\right\}$$

$$= \frac{1}{8}(1-e^{-2\pi}) - \frac{1}{4}\int_0^\pi e^{-2t}\cos(4t)\,dt.$$

since $\cos(4\pi) = \cos(0) = 1$. Note the integral we're trying to compute has reappeared with a $-1/4$ in front of it, so bringing it to the other side by adding gives

$$\int_0^\pi e^{-2t}\cos(4t)\,dt + \frac{1}{4}\int_0^\pi e^{-2t}\cos(4t)\,dt = \frac{1}{8}(1-e^{-2\pi})$$

$$= \frac{5}{4}\int_0^\pi e^{-2t}\cos(4t)\,dt$$

so

$$\int_0^\pi e^{-2t}\cos(4t)\,dt = \frac{1}{10}(1-e^{-2\pi})$$

We now have both integrals we need, so

$$\langle V^2\rangle = \frac{V_0^2}{2\pi}\left[\frac{1}{2}(1-e^{-2\pi}) - \frac{1}{10}(1-e^{-2\pi})\right]$$

$$= \frac{V_0^2}{5\pi}(1-e^{-2\pi})$$

Therefore,

$$V_{RMS} = \sqrt{\langle V^2\rangle} = \boxed{V_0\sqrt{\frac{1-e^{-2\pi}}{5\pi}} \approx 0.252\,V_0}$$

so the RMS voltage is slightly more than one-fourth of V_0.

P.3

Newton's method for finding the number a for which $f(a) = 0$ involves making an initial guess x_1, and then getting successively better guesses using the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 1, 2, 3, \ldots$$

The sequence $x_1, x_2, x_3, \ldots$ will get close to a, so we obtain a series of approximations to the zero of $f(x)$. Here, $f(x) = 2x^3 - x^2 + 10x - 5$, so $f'(x) = 6x^2 - 2x + 10$. This gives

$$x_{n+1} = x_n - \frac{2x_n^3 - x_n^2 + 10x_n - 5}{6x_n^2 - 2x_n + 10} = \frac{x_n(6x_n^2 - 2x_n + 10) - (2x_n^3 - x_n^2 + 10x_n - 5)}{6x_n^2 - 2x_n + 10}$$

$$= \frac{4x_n^3 - x_n^2 + 5}{6x_n^2 - 2x_n + 10}$$

We start with $x_1 = 1$; this gives a second estimate of

$$x_2 = \frac{4-1+5}{6-2+10} = 4/7 \approx 0.57143.$$

Putting this into the right-hand side of the above gives x_3 to be

$$x_3 = \frac{4(4/7)^3 - (4/7)^2 + 5}{6(4/7)^2 - 2(4/7) + 10} = \frac{1859}{3710} \approx 0.50108$$

Then using this to get x_4 gives $x_4 \approx 0.50000$. We see that $x_3 - x_4 \approx 0.00108$, so if we decide to stop now, we find an approximate zero of $\boxed{a \approx x_4 \approx 0.50000}$

(In fact, the correct answer is $a = 1/2$; you can see this by factorising $f(x)$ into $2x^3 - x^2 + 10x - 5 = (2x-1)(2x^2+10)$, and so $f(1/2) = 0$ explicitly. The fact that we got an extremely good approximation to 0.5 after only three iterations shows how useful this method can be.)

P.4

This is a simple application of Simpson's rule, and even though we only divide our interval into three regions, we'll see that we get a pretty good approximation of the correct answer...

$a = 1$ and $b = 2$ in this problem, so $\Delta x = \frac{2-1}{4} = 0.25$. This gives the points defining our regions as $x_1 = a = 1.00$, $x_2 = a + \Delta x = 1.25$, $x_3 = a + 2\Delta x = 1.50$, $x_4 = a + 3\Delta x = 1.75$ and $x_5 = a + 4\Delta x = b = 2.00$. Since $f(x) = 1/x$, we see that

$$\int_1^2 \frac{dx}{x} \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + f(x_5)\right]$$

$$= \frac{0.25}{3}\left[\frac{1}{1.00} + \frac{4}{1.25} + \overset{4}{\frac{2}{1.5}} + \frac{4}{1.75} + \frac{1}{2.00}\right]$$

$$= \frac{1747}{2520} \approx \boxed{0.69325}$$

The correct value $\ln(2) \approx 0.69315$, so we see that we're already within 0.0010 of the real answer, or about 0.015% away. So even this crude application of Simpsons rule with four regions gives us $\boxed{\text{an extremely good approximation!}}$