Solutions to EE106 Exam, Winter 2014

P.1

(a) To check convergence of a series of the form $\sum_{n=0}^{\infty} s_n$, we need to determine what the value of $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ is; if it's undefined or larger than 1, the series is divergent, but if it's less than 1, it converges.

For this series, $s_n = \frac{1}{x^n}$, so we see

$$\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to\infty}\left|\frac{1/x^{n+1}}{1/x^n}\right| = \lim_{n\to\infty}\frac{1}{|x|} = \frac{1}{|x|}.$$

Thus, if $\frac{1}{|x|} < 1$, it converges, which is equivalent to $\boxed{|x| > 1}$ [5pts.]

(b) Every factor in the limited function ("limitand"?) converges as $x \to -2$, so the limit is simply the value of the function at $x = -2$:

$$\lim_{x\to -2}\left(\frac{x-2}{x^2-1}\tan(x)\right) = \frac{(-2)-2}{(-2)^2-1}\tan(-2) = \frac{-4}{3}\tan(-2) = \boxed{\frac{4}{3}\tan(2)}$$ [5pts]

(c) The definition of $f'(x)$, the derivative of $f(x)$, is

$$\boxed{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$ [5pts]

(assuming $f(x)$ is continuous at x, of course.) Thus, to find the derivative of $f(x) = 2(x-5)^2$, we need $f(x+h) - f(x)$:

$$\begin{aligned} f(x+h) - f(x) &= 2(x+h-5)^2 - 2(x-5)^2 \\ &= 2(x^2+h^2+25+2xh-10x-10h) - 2(x^2-10x+25) \\ &= (4x-20)h + 2h^2 \end{aligned}$$

Thus, $\frac{f(x+h)-f(x)}{h} = 4x - 20 + 2h$, giving

$$f'(x) = \lim_{h\to 0}\left[4x-20+2h\right] = \boxed{4x-20}$$ [5pts]

(d) To find the critical points, we look for those points a where $f'(a) = 0$. Here, $f'(x) = (x-3)^3 + 3x(x-3)^2 = (4x-3)(x-3)^2$, so $f'(a) = 0 \Rightarrow (4a-3)(a-3)^2 = 0$. Thus, the critical points occur at $\boxed{3/4 \text{ and } 3.}$ [2pts each]

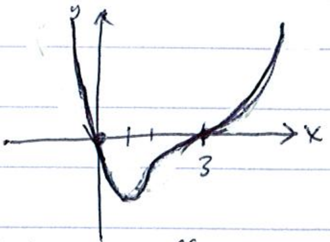
To determine if these are maxima, minima or neither, we need to look at $f''(a)$; if it's positive, we have a minimum. Negative means maximum and 0 means neither. First, the second derivative:

$$\begin{aligned} f''(x) = \frac{d}{dx}\left[(4x-3)(x-3)^2\right] &= 4(x-3)^2 + 2(4x-3)(x-3) \\ &= (12x-18)(x-3) \end{aligned}$$

At $3/4$, we have $f''(3/4) = \left(12\cdot\frac{3}{4} - 18\right)\left(\frac{3}{4}-3\right) = (-9)(-9/4) = 81/4 > 0$

so $\boxed{x = 3/4 \text{ is a minimum.}}$ However, we see that $f(3) = 0$, so [3pts]

$\boxed{x=3 \text{ is neither a max nor a min.}}$ A plot confirms this: inflection [3pts]



pts at $x=3/2$ and $x=3$ give it the correct shape to confirm that $x=3$ is neither max nor min, but $x=3/4$ is definitely a min.

(e) All differential eq's of this type – first order, homogeneous, constant-coefficient – have a sol'n of the form e^{rx}. Putting this in gives $\frac{dy}{dx} = re^{rx} = 10y = 10e^{rx}$, so $r=10$. Thus, a solution to this DE is $\boxed{e^{10x}}$ [5pts]

(f) The area under the curve from 1 to 10 is simply the integral of $f(x)$ over this interval, i.e.

$$\text{area} = \int_1^{10} f(x)\,dx = \int_1^{10}\left(\frac{2}{x} - \frac{1}{x^2}\right) = 2\ln(x) + \frac{1}{x}\Big|_1^{10}$$

$$= \left(2\ln(10) + \frac{1}{10}\right) - \left(2\ln(1) + 1\right) = \boxed{2\ln(10) - \frac{9}{10}}$$ [10pts]

(g) The $4+x^2$ in the denominator strongly suggests a tangent; in fact, a^2+x^2 implies that a substitution $x = a\tan\theta$ [5pts] should work. If we try this, we see

$$\int \frac{dx}{4+x^2} = \int \frac{d(2\tan\theta)}{4+(2\tan\theta)^2} = \int \frac{2\sec^2\theta\,d\theta}{4+4\tan^2\theta}$$

$$= \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta}$$

One of the trig identities that the student should know is $1+\tan^2\theta = \sec^2\theta$, so this gives $\frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$, where C is an arbitrary constant. Now, if $x = 2\tan\theta$, then $\theta = \arctan\left(\frac{x}{2}\right)$, so we get the final answer of $\boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$ [5pts]

P.2

l'Hôpital's rule says that, in evaluating the limit of $\frac{f(x)}{g(x)}$ as $x \to a$, that $\boxed{\text{if } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} \frac{1}{f(x)} = \lim_{x\to a}\frac{1}{g(x)} = 0,}$ ~~then~~

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$ [5pts]

if the second limit exists.

The limit we have here satisfies this criteria: $f(x) = (\sinh(3x))^2$ and

$g(x) = x^2$ both go to 0 as $x \to 0$, so since $f'(x) = 6\sinh(3x)\cosh(3x)$ and $g'(x) = 2x$, we have

$$\lim_{x\to 0} \frac{(\sinh(3x))^2}{x^2} = \lim_{x\to 0} \frac{6\sinh(3x)\cosh(3x)}{2x} = 3\lim_{x\to 0}\frac{\sinh(3x)}{x}$$

since $\lim_{x\to 0}\cosh(3x) = 1$. This limit is still of indeterminate form, so we apply l'Hôpital's rule again:

$$\lim_{x\to 0}\frac{(\sinh(3x))^2}{x^2} = 3\lim_{x\to 0}\frac{\sinh(3x)}{x} = 3\lim_{x\to 0}\frac{3\cosh(3x)}{1} \boxed{= 9}$$ [5pts]

(b) Taylor's theorem says that if $f(x)$ is "nicely-behaved" at $x=a$ (i.e. all its derivatives are defined at a), then $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$ for values of x where this series converges. (Here, $f^{(n)}(a) = \frac{d^n f}{dx^n}(a)$). [5pts]

For the function given, we need its first three derivatives to get its first four terms. These are easy to compute:

$$f(x) = \sqrt{1+x} \Rightarrow f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f''(x) = -\frac{1}{4(1+x)^{3/2}} \Rightarrow f'''(x) = \frac{3}{8(1+x)^{5/2}}.$$

We specifically expand around $a = 0$, so we see $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = -1/4$ and $f'''(0) = 3/8$. Thus,

$$\sqrt{1+x} = \frac{1}{0!}x^0 + \frac{1/2}{1!}x^1 + \frac{-1/4}{2!}x^2 + \frac{3/8}{3!}x^3 + \dots$$

$$= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots}$$ [10pts]

P.3

(a) This is an extremely common 2^{nd} order DE of the form $\frac{d^2y}{dx^2} + a^2y = 0$, and we know that all such DEs have two sol'ns of the form $\sin(ax)$ and $\cos(ax)$. Here, $a^2 = 25$ is the coefficient of y, so $a = 5$ gives us $\boxed{\sin(5x) \text{ and } \cos(5x)}$ as the sol'ns we want. [5pts each]

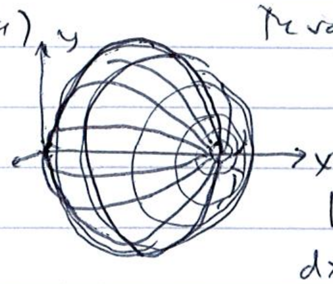
(b) As we saw in P.1(e), e^{rt} is the sol'n, with $r = -k$. So a general soln has the form $N(t) = Ae^{-kt}$ for some constant A. However, since $A = N(0)$, $A = N_0$ is the initial concentration of ^{244}Pu, so the full sol'n is $\boxed{N(t) = N_0e^{-kt}}$. [10pts.]

Now, the half-life of a material is the time τ it takes for it to be reduced to half its initial concentration, i.e. $N(\tau) = N_0/2$. Thus, $N_0e^{-k\tau} = N_0/2 \Rightarrow e^{-k\tau} = \frac{1}{2} \Rightarrow -k\tau = \ln(1/2) = -\ln 2$

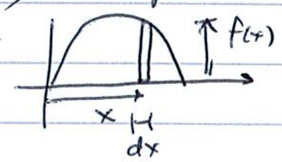
so $\tau = \frac{\ln 2}{k}$. Using the value of k given, we find

$\boxed{\tau = 8\times 10^7 y = 80 \text{ million years.}}$ [5pts]

P.4

(a) 

The volume of revolution looks something like the picture to the left. The key point is that each slice of the solid



located at position x is dx thick and has a radius of $f(x)$. Since each slice is a circular disk, the volume of each slice is $\pi(f(x))^2 dx$, in this case $\pi x^2(1-x)^2 dx = \pi(x^2-2x^3+x^4)dx$. The total volume is thus the integral of this from $x=0$ to $x=1$, i.e.

$$V = \int_0^1 \pi (f(x))^2 dx = \pi \int_0^1 (x^2-2x^3+x^4)\,dx$$

$$= \pi \left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) = \boxed{\frac{\pi}{30}.}$$ [10pts.]

(b) Integration by parts is extremely useful if we happen to have integrals of the form $\int u\,dv$: since $d(uv) = du\,v + u\,dv$ by the product rule, it's easily shown that

$$\boxed{\int u\,dv = uv - \int v\,du}$$ [5pts]

(or, for definite integrals, $\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$.)

Now, the integral given is an integral which is the product of two functions, x and $\sin(x)$. We could take $u=\sin(x)$ and $dv = x\,dx$, but you quickly see this will not get us anywhere. Thus, try $u = x$ and $dv = \sin(x)\,dx$. $du = dx$ follows easily, and since the integral of $\sin(x)$ is $-\cos(x)$, $v = -\cos(x)$. Thus, our formula above gives

$$\int x\sin(x)\,dx = \int x\,d(-\cos(x)) = (x)(-\cos(x)) - \int(-\cos(x))(dx)$$

$$= -x\cos(x) + \int \cos(x)\,dx.$$

The remaining integral is easy to do, so we have our final answer:

$$\int x\sin(x)\,dx = \boxed{-x\cos(x) + \sin(x) + C}$$ [10pts]

where C is an arbitrary constant.