EE106 – Engineering Mathematics I

Problem Set 6

Due in tutorial on Thursday, 20 November 2014

1. A radioactive element has a half-life of 33150 years. The number of atoms in a sample of this element is given at time t by a function $N(t)$ that satisfies the differential equation (DE)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN$$

where k is the decay constant of the element.

   (a) If the initial number of atoms in the sample is N_0, show that $N(t) = N_0 e^{-kt}$.

   (b) Find the time it takes for 33% of the atoms to decay.

2. Suppose that $f(x)$ satisfies the DE

$$2f''(x) + 5f'(x) - 3f(x) = 0.$$

   (a) Show that

$$f(x) = Ae^{x/2} + Be^{-3x}$$

   is a solution to the DE for any choice of the constants A and B.

   (b) Find the specific constants such that $f(0) = 1$ and $f'(0) = -1$.

3. Write down two independent solutions to the DE

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 196y(t) = 0.$$

4. Show that if a is a positive constant, then

$$y(x) = \frac{\sin(x) + (a-1)\cos(x)}{a^2 - 2a + 2}$$

is a solution to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + ay = \cos(x).$$

(**Comment:** This solution illustrates the phenomenon of *resonance*. As the constant a gets closer to 1, the maximum value of $y(x)$ increases, and if a moves away from 1, the maximum value decreases.)