CHAPTER IV
Miscellaneous topics

§ 1. Volumes, surfaces and curves

W E shall use this chapter to discuss an assorted list of topics most of them applications of calculus. We begin with an important use of integration to calculate *volumes*, *surface areas* and *lengths of curves* rather than just the *area* under a curve. We shall deal first with volumes. To this end we introduce what is called a *volume of revolution* and then calculate its volume.

§§ 1.1 Volumes of revolution and their volumes

A volume of revolution is made by rotating a curve through one complete revolution—i.e. 2π radians—about an axis. In fig. 38 we show what is needed to understand what is going on. The first part of the figure simply show the graph of the curve $f(x)$ while the second illustration shows the solid obtained by rotating this curve once about the x-axis. Such a solid is called a *volume of revolution*. We shall now see how to calculate the volume of this solid.

All we have to do is to divide the solid up into cylinders or disks: fig. 38 shows a typical disk, with *centre x*, *radius $f(x)$* and thickness dx.
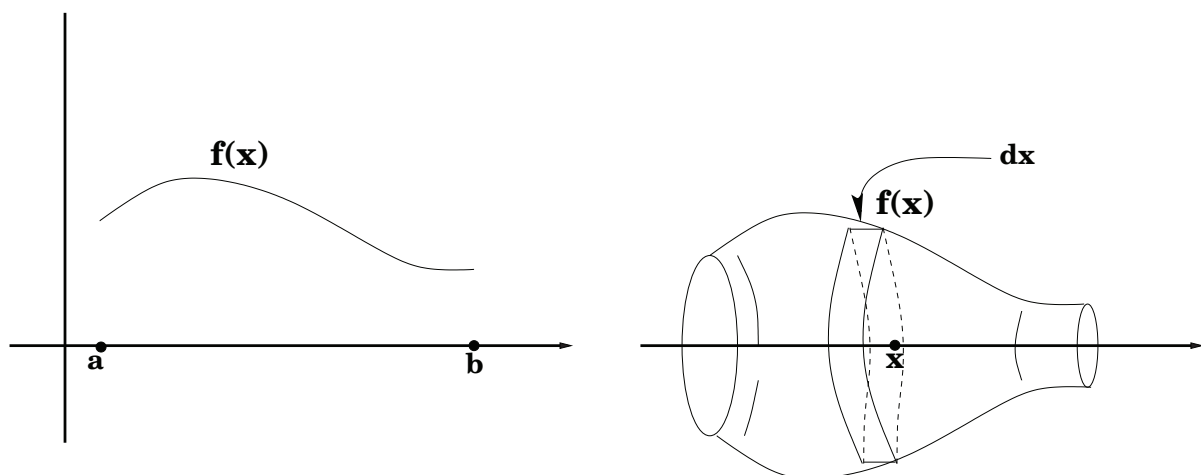

Fig. 38: A volume of revolution made from the function $f(x)$

Now a disk of radius r and thickness h is just a cylinder and so has volume

$$\pi r^2 h \tag{4.1}$$

Hence the disk shown in fig. 38 has volume

$$\pi (f(x))^2 dx \tag{4.2}$$

The entire volume V of the solid will be obtained by summing over all disks as their thickness dx tends to zero; but this is just the integral

$$\int_a^b \pi f^2(x)\, dx \tag{4.3}$$

where a and b are the points where the curve $f(x)$ begins and ends. Thus we have a formula for the volume V of the surface of revolution namely

$$V = \int_a^b \pi f^2(x)\, dx \tag{4.4}$$

Let us use this formula in an actual calculation.

Example *The volume of an ellipsoid of revolution*

An ellipse centered at the origin has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.5}$$
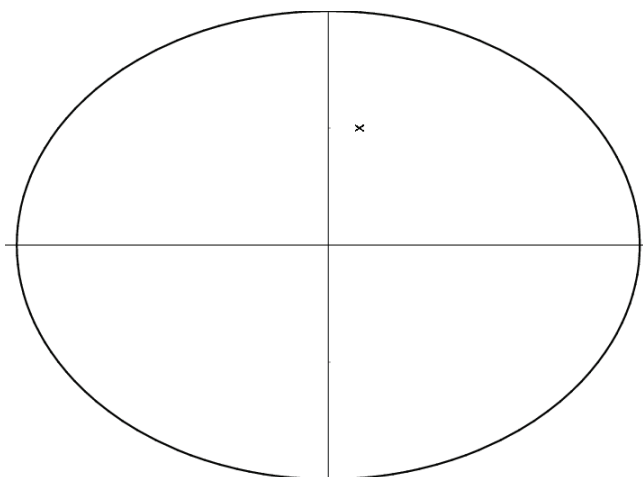
and is shown in fig. 39.

Fig. 39:   The ellipse $x^2/a^2 + y^2/b^2 = 1$

If we take *half* of this ellipse—as shown in fig. 40—and rotate it about the x axis we obtain a solid called an *ellipsoid of revolution*.
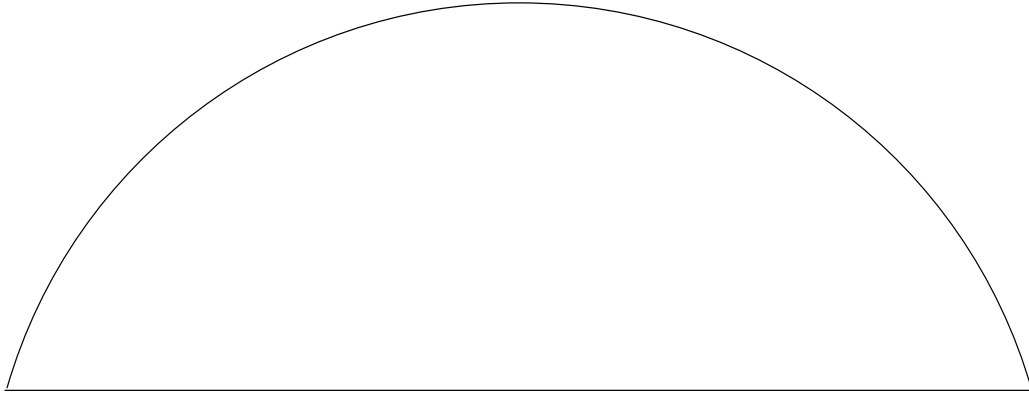

Fig. 40: The upper half of the ellipse $x^2/a^2 + y^2/b^2 = 1$

Now we need to find the appropriate function $f(x)$ so that we can use formula 4.4; but $f(x)$ is just the quantity y so this will come at once from the equation of the ellipse: We have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right) \tag{4.6}$$

$$\Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$$

Hence $f(x)$ is given by the equation

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}} \tag{4.7}$$

Now we can use formula 4.4 to compute the volume V of this ellipsoid. This gives us that

$$V = \int_a^b \pi f^2(x)\, dx$$

$$= \pi \int_{-a}^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \pi b^2 \left[x - \frac{x^3}{3a^2} \right]_{-a}^a \tag{4.8}$$

$$= \pi b^2 \left[a - \frac{a^3}{3a^2} - (-a) + \frac{(-a)^3}{3a^2} \right]$$

$$= \pi b^2 \left[\frac{4a}{3} \right]$$

So the volume of the ellipsoid of revolution is given by

$$V = \frac{4\pi ab^2}{3} \qquad (4.9)$$

Notice that, since a circle is a special case of an ellipse, we should be able to reproduce the formula for the volume of a sphere by making our ellipse a circle. This will provide a check on our calculation and so we shall do it. All we have to do is to set

$$b = a \qquad (4.10)$$

and then the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$
$$\Rightarrow x^2 + y^2 = a^2 \qquad (4.11)$$

i.e. the ellipse has a become a circle of radius a. Such a circle would, if we rotated its upper half, produce a sphere of radius a which we know has a volume

$$\frac{4\pi a^3}{3} \qquad (4.12)$$

But if we set $b = a$ in the formula 4.9 for the volume of our ellipsoid we obtain

$$\frac{4\pi a^3}{3} \qquad (4.13)$$

which is indeed the volume of the sphere, and our check has succeeded.

§§ **1.2 Surfaces of revolution and their areas**

We can also use integration to find the area of the *surface* of one of these solids. Such a surface is called, not surprisingly, a *surface of revolution.*

Now to calculate the area of this surface one can again use disks but this time they the disks are not small cylinders of height dx but *skew disks* of *slanted height dl*. An example of a skew disk is shown in fig. 41.
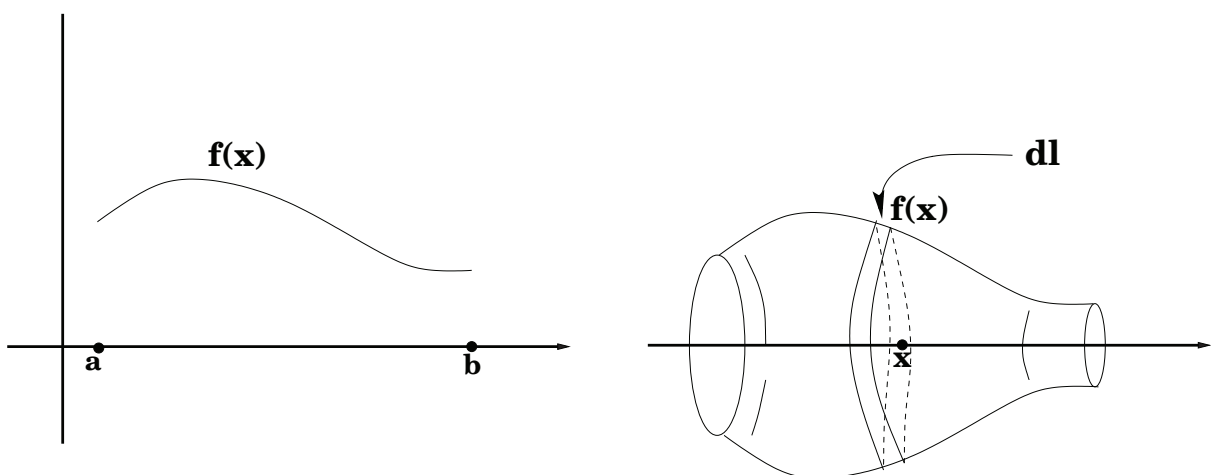

Fig. 41: A surface of revolution and its division into *skew* disks

The reader should carefully compare figs. 38 and 41 until he or she notices that the disk of 38 have *straight sides* of height dx while those of 41 have *slanted sides* of length dl.

The important point is that dl is, in general, not equal to dx because dl points along the direction of the tangent to f at x while dx is always parallel to the x-axis. But if the tangent makes an angle θ with the x-axis at the point x then, by definition of the derivative, we know that

$$f'(x) = \tan(\theta) \tag{4.14}$$

It is also true that dx and dl are at an angle θ relative to one another so that we have

$$dx = dl\cos(\theta) \tag{4.15}$$

Now we are ready to calculate the area: a typical skew disk of radius $f(x)$ and skew side of slanted length dl has *surface area*

$$\begin{aligned} &2\pi f(x)dl \\ &= 2\pi f(x)\frac{dx}{\cos(\theta)}, \quad \text{using 4.15} \end{aligned} \tag{4.16}$$

But we can relate $\cos(\theta)$ to $f'(x)$ by using 4.14. We have the fact that

$$\begin{aligned} 1 + \tan^2(\theta) &= \sec^2(\theta) \\ &= \frac{1}{\cos^2(\theta)} \\ \Rightarrow \frac{1}{\cos(\theta)} &= \sqrt{1 + \tan^2(\theta)} \\ &= \sqrt{1 + f'(x)^2}, \quad \text{using 4.14} \end{aligned} \tag{4.17}$$

So our surface area formula now becomes

$$2\pi f(x)\frac{dx}{\cos(\theta)} = 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \tag{4.18}$$

The final task to get the surface area S of the surface of revolution is to sum over all such skew disks and this gives us an integral. The resulting expression for S

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \tag{4.19}$$

It is now time to look at an example.

Example *The surface area of the ellipsoid*

 The calculation is rather straightforward since we have already had the experience of the volume computation.

Recall that 4.7 says that

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$\Rightarrow f'(x) = b\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{a^2}\right)^{-1/2}\left(\frac{-2x}{a^2}\right)$$

$$= \frac{-bx}{a^2}\frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \tag{4.20}$$

$$\Rightarrow 1 + f'(x)^2 = 1 + \frac{b^2x^2}{a^4}\frac{1}{1 - \frac{x^2}{a^2}}$$

$$= \frac{a^4 + (b^2 - a^2)x^2}{a^4(1 - \frac{x^2}{a^2})}$$

Substituting this information back into the expression 4.19 for the area S gives

$$S = 2\pi \int_{-a}^{a} \frac{b}{a^2}\sqrt{a^4 + (b^2 - a^2)x^2}\, dx \tag{4.21}$$

Now to speed up the calculation[1] we set

$$b = a \tag{4.23}$$

so that we are now computing the surface area of a sphere. With this simplification we find
that

$$S = 2\pi a \int_{-a}^{a} dx$$

$$= 2\pi a\, [x]_{-a}^{a} \tag{4.24}$$

$$= 4\pi a^2$$

---

[1] We don't want to bother the reader with the expression for the surface area when $b \neq a$, but for the
curious we give it anyway in this footnote. If we use the fact that

$$\int \sqrt{1 - x^2}\, dx = \frac{1}{2}\left\{x\sqrt{1 - x^2} + \arcsin(x)\right\}, \quad \text{(integrate by parts twice to get this)}$$

then we find that we can do the integral for S after a suitable change of variable. After doing this, and putting
in the limits, we find that

$$S = 2\pi\left\{b^2 + \frac{a^2b}{\sqrt{a^2 - b^2}}\arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right)\right\} \tag{4.22}$$

and this is indeed the surface area of a sphere of radius a.

We can now turn to the matter of calculating the length of an arbitrary curve.

§§ 1.3 Curves and their lengths

If $f(x)$ is any curve, such as that depicted in fig. 41, then we can easily derive an integral formula for the length of f. If the endpoints of the curve are, as in fig. 41, at $x = a$ and $x = b$ then the length L of the curve is given by

$$L = \int_a^b dl \tag{4.25}$$

where dl is an infinitesimal piece of the curve exactly as it is in 41. But 4.15 gives the relation between dx and dl as

$$dx = dl \cos(\theta) \tag{4.26}$$

and 4.17 tells us that

$$\cos(\theta) = \frac{1}{\sqrt{1 + f'(x)^2}} \tag{4.27}$$

Hence $dl = \sqrt{1 + f'(x)^2}\, dx$ giving us our final formula for the curve length which is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{4.28}$$

We are now ready to do a calculation.

Example *The length of an arc of a circle*

If the circle has radius A then its equation is

$$x^2 + y^2 = A^2$$
$$\Rightarrow y = \sqrt{A^2 - x^2} \tag{4.29}$$

This means we set

$$f(x) = \sqrt{A^2 - x^2} \tag{4.30}$$

and so

$$f'(x) = -\frac{x}{\sqrt{A^2 - x^2}} \tag{4.31}$$

and thus

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \frac{x^2}{A^2 - x^2}}\, dx \\ &= \int_a^b \sqrt{\frac{x^2 - A^2 + x^2}{A^2 - x^2}}\, dx \\ &= A \int_a^b \frac{dx}{\sqrt{A^2 - x^2}} \end{aligned} \tag{4.32}$$

Let us take a circle of unit radius so that $A = 1$ and then we have

$$L = \int_a^b \frac{dx}{\sqrt{1-x^2}} \tag{4.33}$$
$$= [\arcsin(x)]_a^b, \quad \text{using our table of integrals}$$

Finally let us decide to compute the length of a quadrant of this unit circle—cf. fig 42; this means that we must choose the interval $[a, b]$ to be given by
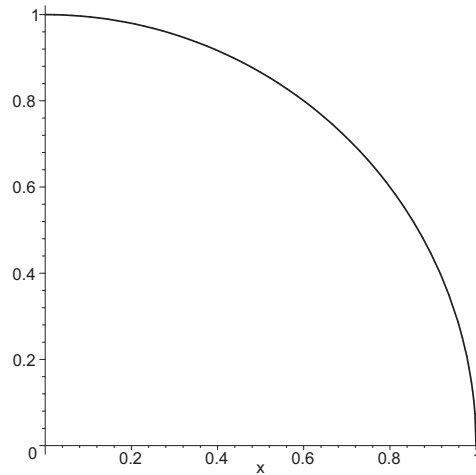
$$[a, b] = [0, 1] \tag{4.34}$$

Fig. 42: A quadrant of a circle of radius 1

We then find that

$$L = [\arcsin(x)]_0^1$$
$$= [\arcsin(1) - \arcsin(0)]$$
$$= \left[\frac{\pi}{2} - 0\right] \tag{4.35}$$
$$= \frac{\pi}{2}$$

and so $L = \pi/2$; a fact which we can easily deduce without integration since the arc is precisely a quarter of the circle's circumference.

Example *Find the length of the curve $f(x) = 1/x + x^3/12$ between $x = 2$ and $x = 5$*
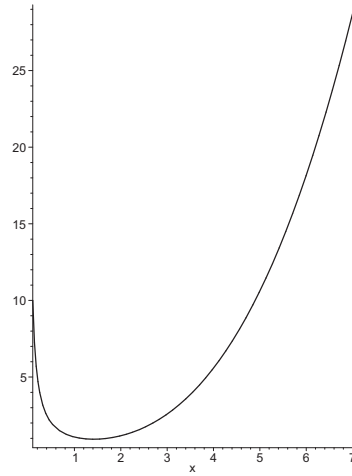
The curve $f(x)$ is shown in fig. 43.

Fig. 43: The curve $f(x) = 1/x + x^3/12$

With

$$f(x) = \frac{1}{x} + \frac{x^3}{12} \tag{4.36}$$

we find that

$$f'(x) = -\frac{1}{x^2} + \frac{3x^2}{12} \tag{4.37}$$

and this gives us

$$
\begin{aligned}
1 + f'(x)^2 &= 1 + \left(-\frac{1}{x^2} + \frac{3x^2}{12} \right)^2 \\
&= 1 + \frac{1}{x^4} - \frac{1}{2} + \frac{x^4}{16} \\
&= \frac{1}{x^4} + \frac{1}{2} + \frac{x^4}{16} \\
&= \left(\frac{1}{x^2} + \frac{x^2}{4} \right)^2
\end{aligned} \tag{4.38}
$$

This means that the length L in question is given by

$$
\begin{aligned}
L &= \int_2^5 \sqrt{\left(\frac{1}{x^2} + \frac{x^2}{4}\right)^2}\, dx \\
&= \int_2^5 \left(\frac{1}{x^2} + \frac{x^2}{4}\right)\, dx \\
&= \left[-\frac{1}{x} + \frac{x^3}{12}\right]_2^5 \\
&= \left[-\frac{1}{5} + \frac{5^3}{12} + \frac{1}{2} - \frac{2^3}{12}\right] \\
&= \frac{201}{20}
\end{aligned}
\tag{4.39}
$$

and so L is found.

Example *The length of a piece of a parabola*

For this example we take

$$
f(x) = x^2
\tag{4.40}
$$

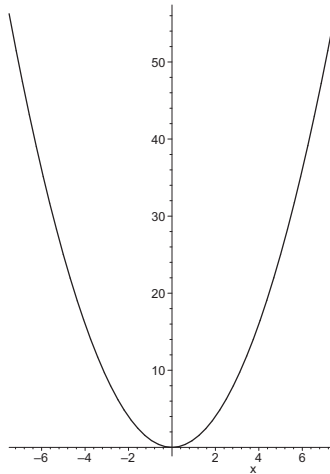which is a parabola and is displayed in fig. 44.

Fig. 44: The parabola $f(x) = x^2$

We readily compute that

$$
\begin{aligned}
f'(x) &= 2x \\
\Rightarrow \sqrt{1 + f'(x)^2} &= \sqrt{1 + 4x^2}
\end{aligned}
\tag{4.41}
$$

Hence the length L of the section of this parabola between $x = a$ and $x = b$ is given by

$$L = \int_a^b \sqrt{1 + 4x^2}\, dx \tag{4.42}$$

If we choose, say, $a = 0$ and $b = 7$ then we find that

$$L = \int_0^7 \sqrt{1 + 4x^2}\, dx \tag{4.43}$$

Now if we simply *accept* that

$$\int \sqrt{1 + 4x^2}\, dx = \frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\operatorname{arcsinh}(2x) \tag{4.44}$$

a fact that we do *not* expect the reader to be able to derive. Then we find that

$$\begin{aligned}
L &= \left[\frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\operatorname{arcsinh}(2x)\right]_0^7 \\
&= \left[\frac{1}{2}7\sqrt{1 + 4 \cdot 49} + \frac{1}{4}\operatorname{arcsinh}(14) - 0\right] \\
&= 49.95821036
\end{aligned} \tag{4.45}$$

where we used a calculator to evaluate the final expression.

§ 2. The Mean value of a function

Suppose we have a function f(x) whose value we measure n times thereby obtaining the series of values

$$f(x_1), f(x_2), \ldots f(x_n) \tag{4.46}$$

The mean, or average, of these n measurements is

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \tag{4.47}$$

We shall denote [2] this mean by $< f >$ so that

$$< f >= \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \tag{4.48}$$

---

[2] The mean is also sometimes denoted by $\bar{f}$.

If n becomes very large—or indeed tends to infinity—then the resulting mean, which we shall still denote by $< f >$, can be expressed as an integral. We shall now explain this.

Suppose that the possible values of x all lie within the interval $[a, b]$ and suppose that we consider, for a moment the integral of f over this interval. Returning to 3.3 for the definition of an integral we have[3]

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\Delta x \tag{4.49}$$

But since the interval $[a, b]$ is divided into precisely n pieces, each of size Δx, we have

$$n\Delta x = b - a$$
$$\Rightarrow \Delta x = \frac{(b - a)}{n} \tag{4.50}$$

Substituting this into our expression for the integral gives

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\frac{(b - a)}{n}$$
$$= \lim_{n \to \infty} \frac{\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)}{n}(b - a) \tag{4.51}$$
$$= < f > (b - a)$$

In other words we have shown that the mean $< f >$ of f over the interval $[a, b]$ is just $\int_a^b f\,dx$ divided by the length of the interval; that is

$$< f > = \frac{\int_a^b f(x)\,dx}{b - a} \tag{4.52}$$

The importance of this way of defining the mean is that it is very well suited to quantities that vary *continuously* with time such as the voltage output from an electrical device. Here is an example.

Example *The mean voltage produced by an oscillator*

An oscillator produces a current output $I(t)$ given by

$$I(t) = a\cos(10\omega t) + b\cos(\omega t) \tag{4.53}$$

where t stands for time—cf. fig. 45.

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[3] Note when comparing with 3.3 that we have used n instead of $n - 1$, and written Δx for $(x_{i+1} - x_i)$
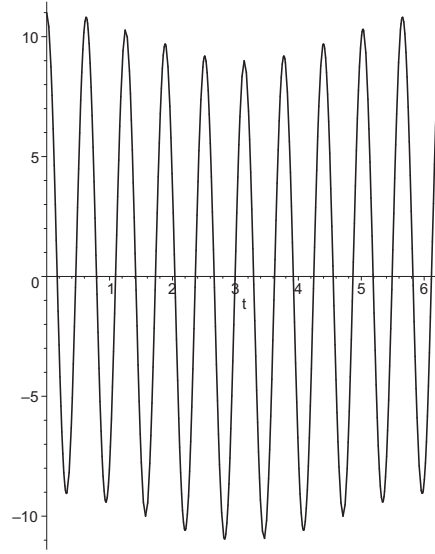
Fig. 45: The oscillator current $I(t) = a\cos(10\omega t) + b\cos(\omega t)$ for $a = 10$, $b = 1$, $\omega = 1$

Notice that $I(T)$ is a sum of a very rapidly varying term $a\cos(10\omega t)$ and a term of slower variation $b\cos(\omega t)$. The term of rapid variation has a periodic time T_{rapid} given by

$$T_{rapid} = \frac{2\pi}{10\omega} \tag{4.54}$$

and we shall now calculate the mean of $I(t)$ over this period. We have

$$< I > = \frac{1}{T_{rapid}} \int_0^{T_{rapid}} (a\cos(10\omega t) + b\cos(\omega t)) \, dt, \quad T_{rapid} = \frac{2\pi}{10\omega} \tag{4.55}$$

But

$$
\begin{aligned}
\int_0^{T_{rapid}} (a\cos(10\omega t) + b\cos(\omega t)) \, dt &= \left[\frac{a}{10\omega}\sin(10\omega t) + \frac{b}{\omega}\sin(\omega t) \right]_0^{T_{rapid}} \\
&= \left[\frac{a}{10\omega}\sin(10\omega T_{rapid}) + \frac{b}{\omega}\sin(\omega T_{rapid}) - 0 \right] \\
&= \frac{a}{10\omega}\sin\left(10\omega\frac{2\pi}{10\omega}\right) + \frac{b}{\omega}\sin\left(\omega\frac{2\pi}{10\omega}\right) \quad (4.56) \\
&= \frac{a}{10\omega}\sin(2\pi) + \frac{b}{\omega}\sin\left(\frac{2\pi}{10}\right) \\
&= 0 + \frac{b}{\omega}\sin\left(\frac{2\pi}{10}\right)
\end{aligned}
$$

Hence we find that

$$< I > = \frac{1}{T_{rapid}} \frac{b}{\omega} \sin\left(\frac{2\pi}{10}\right)$$

$$= \frac{10\omega}{2\pi} \frac{b}{\omega} \sin\left(\frac{2\pi}{10}\right) \tag{4.57}$$

i.e. $$< I > = \frac{10b}{2\pi} \sin\left(\frac{2\pi}{10}\right)$$

Note that the periodic time T of the *combined* system of both terms is just the periodic time of the slowest varying term; so in this case it is

$$T = \frac{2\pi}{\omega} \tag{4.58}$$

and if we calculate the mean $< I >$ over this time T we shall find that

$$< I >= 0 \tag{4.59}$$

because, during this time T, the graph of $I(T)$ spends exactly the same amount of time above the t axis as below (cf. fig. 45) and this causes the integral $\int_0^T I(t)\,dt$ to vanish.

This vanishing of the mean of a function is illustrated in the next example.

Example *The mean of the simple sinusoidal voltage $V(t) = a\sin(\omega t)$*

In this example we have a variable voltage which $V(t)$ is just a sine wave and so has periodic time $T = 2\pi/\omega$ and we can see at once that its mean is zero. Calculating the mean we have

$$< V > = \frac{1}{T} \int_0^T a\sin(\omega t)\,dt, \quad T = \frac{2\pi}{\omega}$$

$$= -\frac{1}{T} \left[\frac{a\cos(\omega t)}{\omega}\right]_0^T$$

$$= -\frac{1}{T} \left[\frac{a\cos(\omega T)}{\omega} - \frac{a\cos(0)}{\omega}\right] \tag{4.60}$$

$$= -\frac{\omega}{2\pi} \left[\frac{a\cos(2\pi)}{\omega} - \frac{a}{\omega}\right]$$

$$= -\frac{\omega}{2\pi} \left[\frac{a}{\omega} - \frac{a}{\omega}\right]$$

i.e. $$< V > = 0$$

§ 3. The Root Mean Square or RMS value of a function

Since the mean of a voltage such as $V(t) = a\sin(\omega t)$ vanishes and so $< V >$ tells us little about such a voltage it is useful to introduce another related mean which gives more information about $V(t)$. This is called the *Root Mean Square value* or the *RMS value* of the function in question.

If f is any function then we shall denotes its RMS value on an interval $[a, b]$ by

$$f_{RMS} \tag{4.61}$$

where the definition of f_{RMS} is that

$$f_{RMS} = \sqrt{< f^2 >}$$

So f_{RMS}^2 is the average of f^2 rather than f. The important point to note then is that since f^2 is *always positive*—being the square of something—then f_{RMS} will never be zero unless f itself is identically zero which is a trivial case.

In any case we have

$$f_{RMS} = \sqrt{< f^2 >} \tag{4.62}$$

and it is time to do an example.

Example *The RMS value of the voltage $V(t) = V_0 \sin(\omega t)$ over its period T*

First we note from our previous work above that the period $T = 2\pi/\omega$ and then we calculate

$$< V^2 > \tag{4.63}$$

This is given by

$$
\begin{aligned}
< V^2 > &= \frac{1}{T} \int_0^T V^2(t)\, dt \\
&= \frac{1}{T} \int_0^T V_0^2 \sin^2(\omega t)\, dt \\
&= \frac{1}{T} \int_0^T V_0^2 \frac{1}{2} \left(1 + \cos(2\omega t)\right)\, dt, \quad \text{using } \sin^2(t) = \frac{1}{2}\left(1 + \cos(2t)\right) \\
&= \frac{1}{T} \frac{V_0^2}{2} \left[t + \frac{1}{2\omega} \sin(2\omega t)\right]_0^T \\
&= \frac{1}{T} \frac{V_0^2}{2} \left[T + \frac{1}{2\omega} \sin(2\omega T) - 0\right] \\
&= \frac{\omega}{2\pi} \frac{V_0^2}{2} \left[\frac{2\pi}{\omega} + \frac{1}{2\omega} \sin(4\pi)\right] \\
&= \frac{\omega}{2\pi} \frac{V_0^2}{2} \left[\frac{2\pi}{\omega} + 0\right] \\
&= \frac{V_0^2}{2}
\end{aligned}
\tag{4.64}
$$

So $V_{RMS} = \sqrt{<V^2>}$ is given by

$$V_{RMS} = \frac{V_0}{\sqrt{2}} \tag{4.65}$$

This is a well known result in electrical engineering.

The reader should note that, since $V(t) = V_0 \sin(\omega t)$, then the *largest value* that $V(t)$ can have is when $\sin(\omega t)$ has its largest value which is unity; this in turn means that the maximum value of $V(t)$ is V_0 and V_0 is often called the *peak voltage*. So it is both common and useful to think of V_{RMS} as being $1/\sqrt{2}$ times the peak voltage; it is worth memorising the approximate value of $1/\sqrt{2}$ so we quote it here

$$\frac{1}{\sqrt{2}} = 0.707\ldots \tag{4.66}$$

We see that V_{RMS} is about 70% of the peak voltage V_0. If a country uses an AC system of voltage supply the voltage usually quoted is the RMS value—230 volts for Ireland.

§ 4. Numerical Integration

We have seen already that many integrals are hard to do in closed from and so have to be done numerically. there are many techniques for doing this. Two simple ones which are often introduced at the beginning of a discussion on numerical integration are the *trapezoidal rule* and *Simpson's rule*; the latter being superior to the former. We shall have a very brief look at Simpson's rule.

§§ 4.1 Simpson's rule

Suppose we want to evaluate

$$\int_a^b f(x)\,dx \tag{4.67}$$

then we divide the interval $[a, b]$ up into

$$2n \tag{4.68}$$

intervals so that each has size

$$h = \frac{b - a}{2n} \tag{4.69}$$

Now we select $2n + 1$ values of x in $[a, b]$ given by $x_j = a + jh$ for $j = 0, \ldots 2n$, i.e.

$$\begin{aligned}
x_0 &= a \\
x_1 &= a + h \\
x_2 &= a + 2h \\
&\vdots \qquad \vdots \\
x_{2n-1} &= x + 2(n-1)h \\
x_{2n} &= b
\end{aligned} \tag{4.70}$$

With this accomplished *Simpson's rule* gives the following *approximate* value for the integral

$$\int_a^b f(x)\,dx \sim \frac{b-a}{6n}\,\{f(a) + 2(f(x_1) + f(x_3) + \cdots + f(x_{2n-1})) \tag{4.71}$$
$$+\ 4(f(x_2) + f(x_4) + \cdots + f(x_{2n-2})) + f(b)\}$$

and the *error* is proportional to

$$\frac{1}{n^4} \tag{4.72}$$

We simply quote this formula 4.71 as we have no space left to give its (quite straightforward) derivation. We finish by using Simpson's rule in an example.

Example *The integral*

$$\int_0^2 \sqrt{x}\,\sin(x)\,dx \tag{4.73}$$

First if we use a computer package such as *Maple*, *Mathematica* or *Matlab* then they have very good in built numerical integration routines and using Maple we can ask for, say, 10 significant figures of accuracy and then we obtain the result that

$$\int_0^2 \sqrt{x}\,\sin(x)\,dx = 1.235911456 \tag{4.74}$$

Now if we use Simpson's rule, and use Maple to evaluate 4.71 for $n = 10, 20, 30, 40, 50, 60$ and 70, we obtain the set of values

$$
\begin{aligned}
&1.235915368\\
&1.235911716\\
&1.235911508\\
&1.235911474 \qquad\qquad (4.75)\\
&1.235911468\\
&1.235911459\\
&1.235911456
\end{aligned}
$$

and the reader can see the sort of accuracy of Simpson's rule. We obtain the correct answer to 10 significant figures with $n = 70$. It is now easy to treat many more examples and we leave this task to the reader and bring these lectures to a close here.

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CHAPTER III

CHAPTER IV