Overview

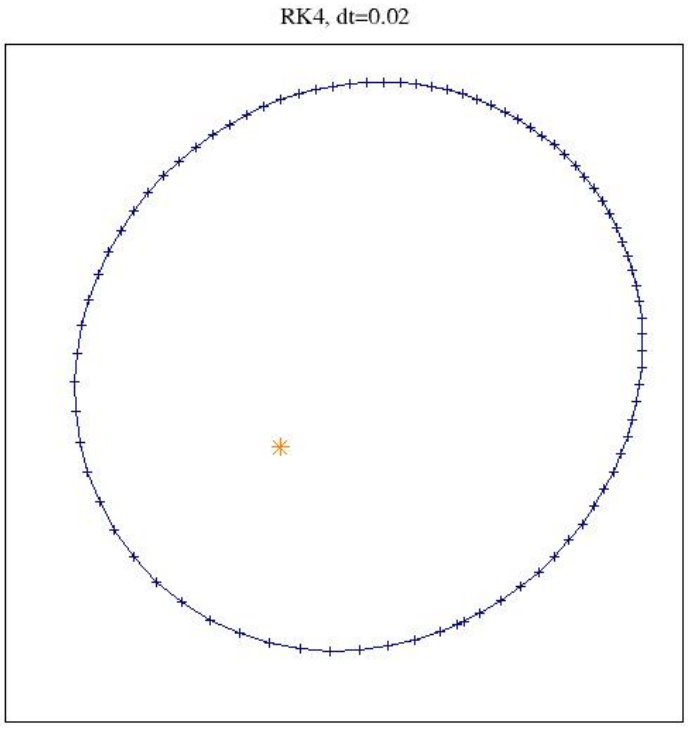
Recap

Runge–Kutta methods: systematic improvement in accuracy

- Midpoint method = second order Runge–Kutta
- Fourth-order Runge–Kutta very widely used
- Higher order used in adaptive stepsize algorithms

Slow vs fast motion

Example: Kepler problem



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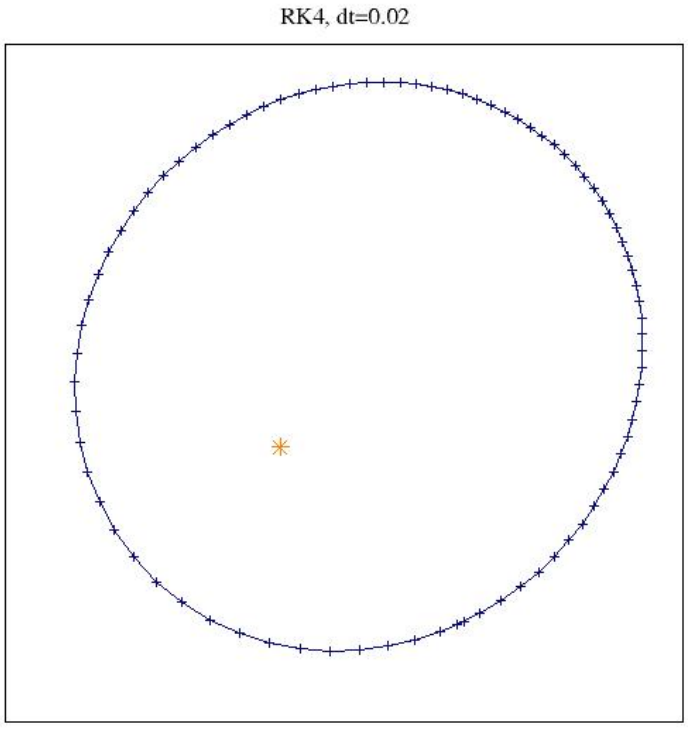

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- $r(t), \theta(t)$ vary rapidly
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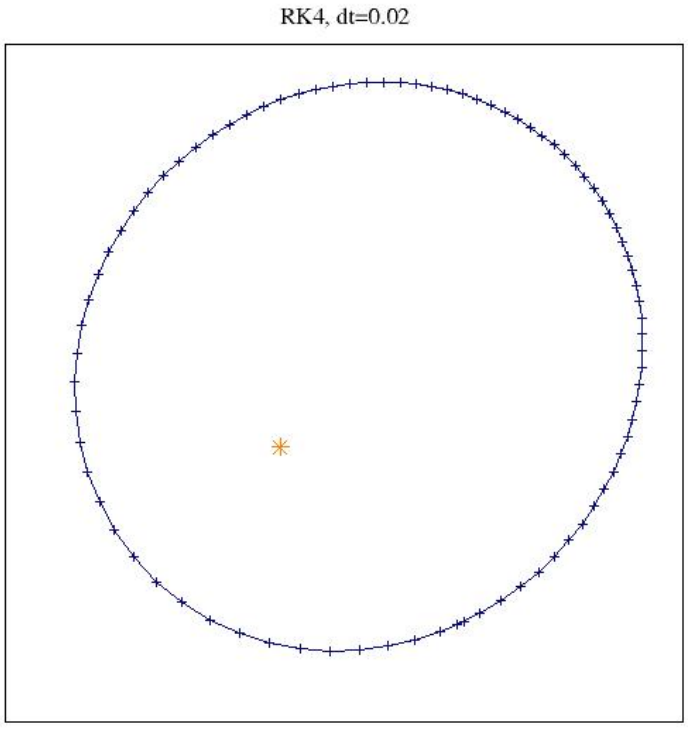

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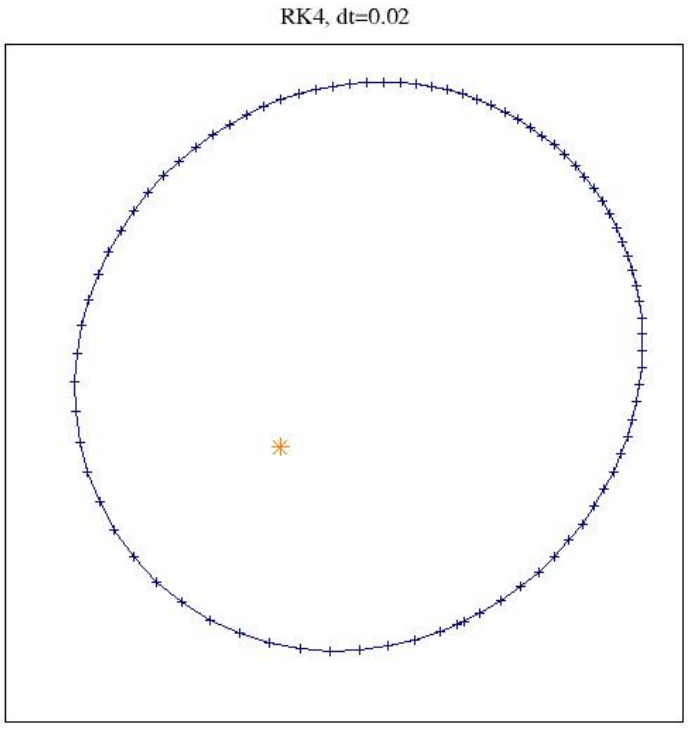

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Solution: Let computer decide what stepsize is needed!

Adaptive stepsize

Let computer decide stepsize depending on what precision we want

- Estimate error at current Δt
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How to estimate the error?

Two approaches:

- Take two steps with $\Delta t/2$, compare with one step with Δt
- Find difference between higher-order step and lower-order (eg RK4 and midpoint)

Adaptive stepsize

What is too small, too big?

- A fixed number is not useful:
  in Kepler problem θ is in radians but r could be in metres!
- Try to keep

$$\left|\frac{\Delta r}{r}\right| \sim \left|\frac{\Delta \theta}{\theta}\right| \sim \Delta$$

— relative tolerance

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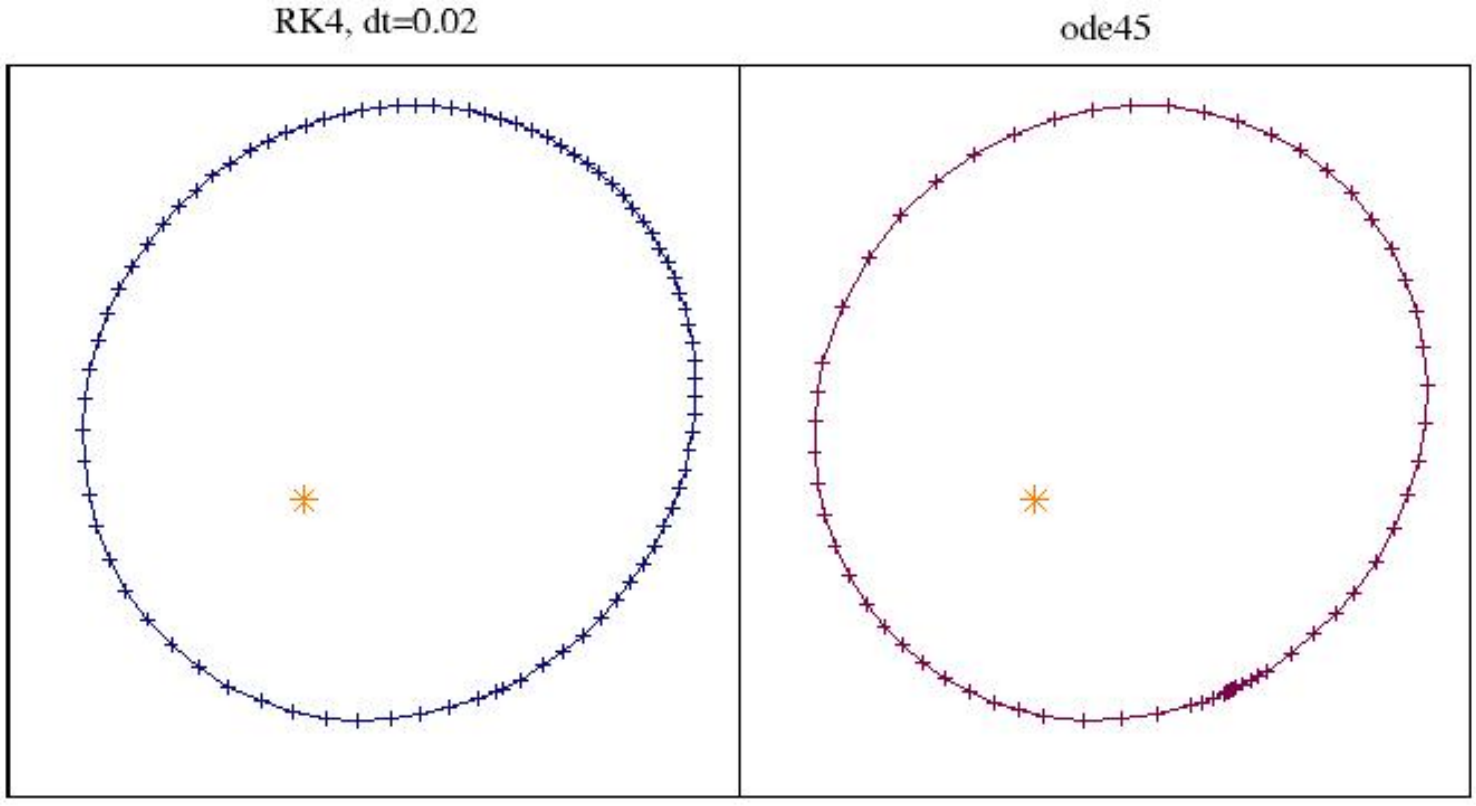
The 'optimal' step size can be estimated using Taylor expansion:

$$\Delta t_{\text{opt}} = \Delta t \left|\frac{\Delta_{\text{rel}}}{\Delta}\right|^{1/N}, \qquad N = \text{order of algorithm}$$

Fixed vs adaptive stepsize



Subjects not covered

- Stiff sets of equations
- Implicit methods
- Richardson extrapolation

Boundary value problems

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We can always reduce this to a set of n coupled first-order ODEs:

$$\begin{aligned} z_1'(x) &= z_2(x) \\ &\vdots \\ z_{n-1}'(x) &= z_n(x) \\ z_n'(x) &= F(z_n(x), \ldots, z_2(x), z_1(x), x) \end{aligned}$$

For example, $y''(x) = \sin(x)y'(x) + (x+1)\ln y(x) + x^2$ becomes

$$\begin{aligned} z_1'(x) &= z_2(x) \\ z_2'(x) &= \sin(x)z_2(x) + (x+1)\ln z_1(x) + x^2 \end{aligned}$$

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n equations: need n conditions to specify problem completely

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- Integrate using Euler–Cromer, Runge–Kutta, Stoer–Bullirsch,...

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Boundary value problem

Everything else!

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Two general methods: shoot, or relax
Linear ODEs can also be transformed to matrix equations.

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Matrix method

- Divide interval $[x_1, x_2]$ into equally spaced, discrete points
- Write down equation for each point using discrete derivatives
- The set of equations can be written as a matrix equation $Ay = b$
- Solve to find y — next semester!

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Higher order: Newton–Raphson

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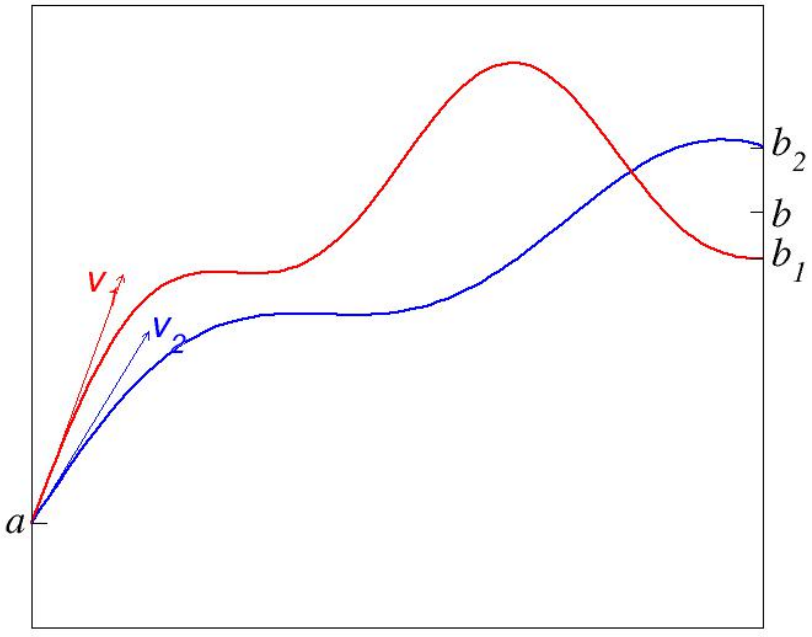

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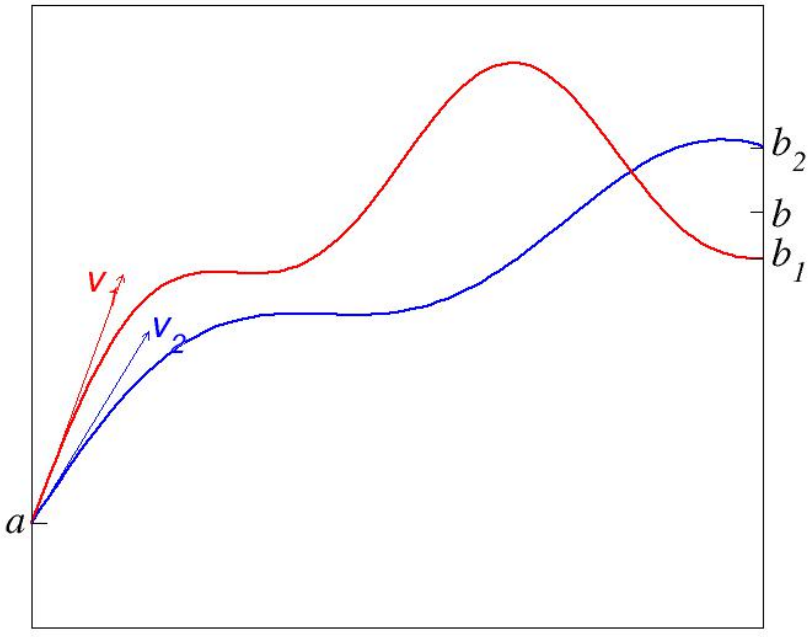

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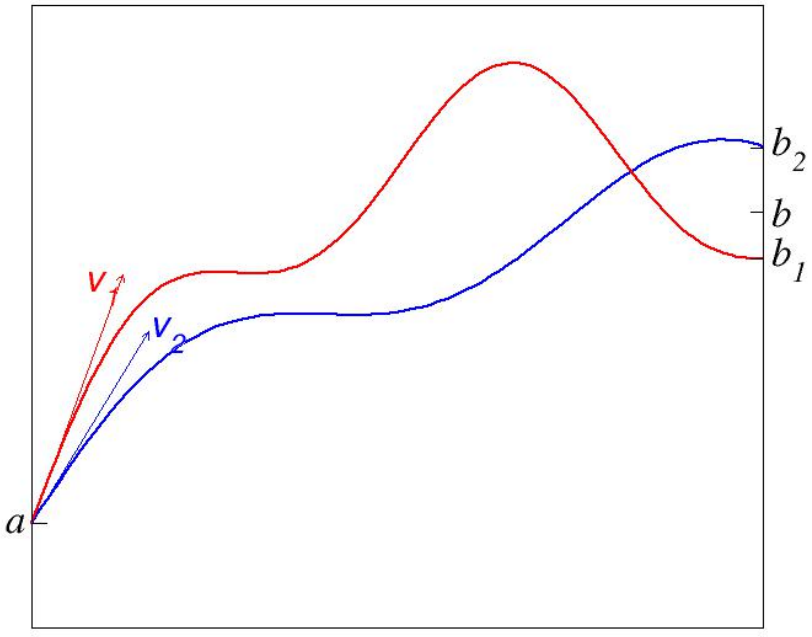

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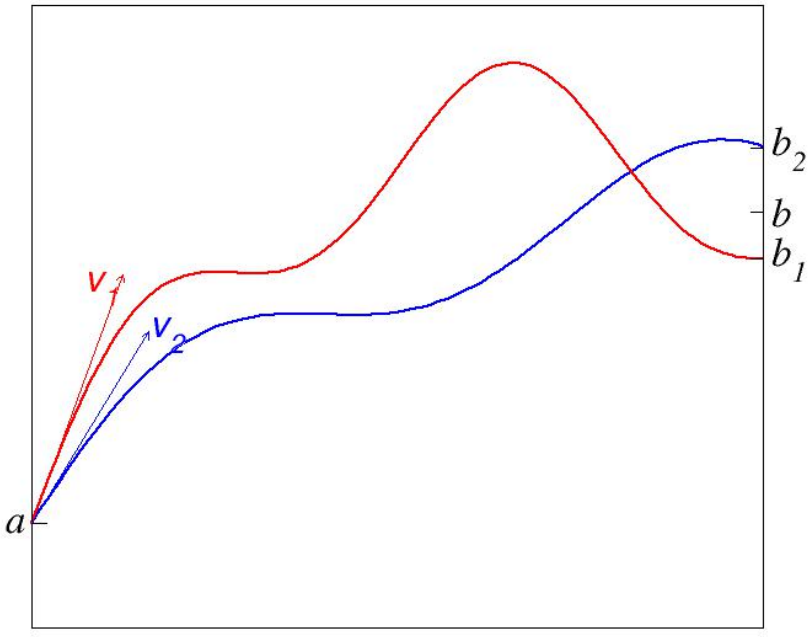

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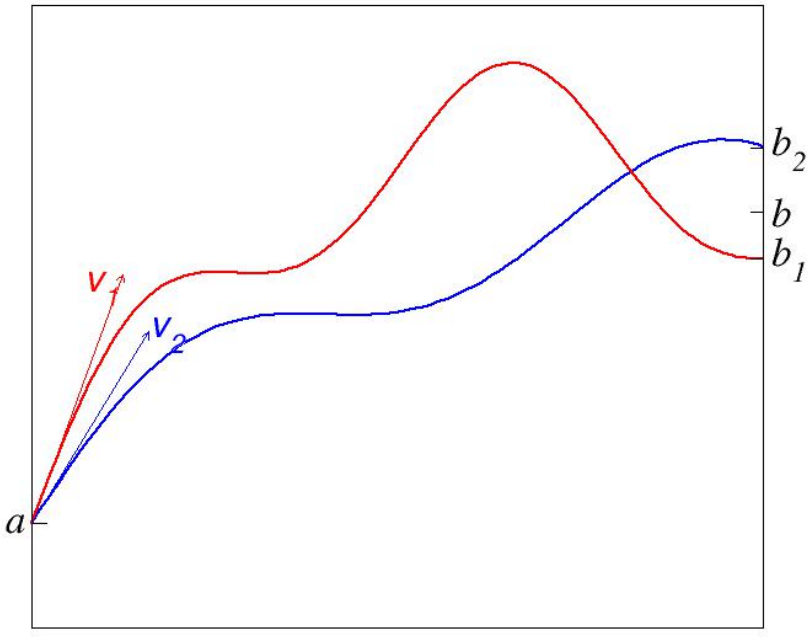

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We can find the value for $y'(x_1)$ using bisection!
This will give us the solution of the boundary value problem.

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- Must (usually) start by guessing a solution!
- Shooting method:
  1. Guess unknown initial values v_i
  2. Solve ODE with these values: $f(x|v_i)$
  3. Find solution at final point x_f
  4. Solve $f(x_f|v_i) - v_f = 0$ — root finding!