Overview

1. Recap: The Euler method
2. Stability
3. Runge–Kutta methods
4. Summary

Recap: The Euler method

We want to solve the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{starting from} \quad y(a) = c \quad x \in [a, b]$$

Algorithm

Set $x_0 = a, y_0 = c$, choose stepsize ε or $N = (b - a)/\varepsilon$
Calculate $v_0 = f(x_0, y_0)$.
Then, for each $n = 1, \ldots, N$ do

1. $x_n = x_{n-1} + \varepsilon$
2. $y_n = y_{n-1} + \varepsilon v_{n-1}$
3. $v_n = f(x_n, y_n)$

The Euler method is accurate up to $\mathcal{O}(\varepsilon)$

Stability 1: 1st order ODEs

To understand stability, let us look at a simple differential equation:

$$\frac{dy}{dx} = -\gamma y = f(x, y)$$

The Euler method

Start with $x_0, y_0, v_0 = f(x_0, y_0)$.

1. $x_{n+1} = x_n + \varepsilon$
2. $y_{n+1} = y_n + \varepsilon f(x_n, y_n)$

Applying the Euler method to our equation, we find

$$y_{n+1} = y_n - \varepsilon\gamma y_n = (1 - \varepsilon\gamma) y_n$$

We know that the solution is an exponential decay.
But if $\varepsilon > 2/\gamma$ we will instead get an increase!

Instability

The Euler method is unstable for $\varepsilon > 2/\gamma$

Stability 2: Hamiltonian equations

Look at a simple second order ODE: $y''(x) = f(x, y)$

Make it into two first order equations as usual

$$\frac{dy}{dx} = z(x) \qquad \frac{dz}{dx} = f(x, y)$$

The Euler method can be written

1. $z_{n+1} = z_n + \varepsilon f(x_n, y_n)$
2. $y_{n+1} = y_n + \varepsilon z_n$

We have already calculated z_{n+1} by the time we update y.
Why not use it instead?

Euler–Cromer

1. $z_{n+1} = z_n + \varepsilon f(x_n, y_n)$
2. $y_{n+1} = y_n + \varepsilon z_{n+1}$

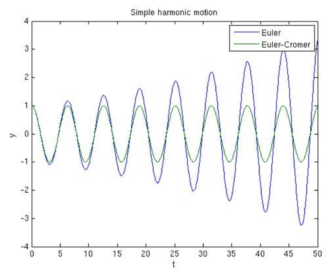
Alternatively: update y first, then z

Euler vs Euler–Cromer

Assertion: Euler–Cromer is better than Euler (and midpoint, next). Why?

The error is $\mathcal{O}(\varepsilon)$, the same as Euler, and worse than midpoint.
That cannot be the answer.

Answer



Euler–Cromer is stable.
Euler and midpoint are not!

Errors in Euler–Cromer do not grow exponentially!

Euler–Cromer is stable because it is a symplectic integrator.
It conserves the 'Hamiltonian'

Harmonic oscillator

$$\ddot{q} + \omega^2 q = 0 \qquad H = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} m\omega^2 q^2$$

Set $\omega = 1, m = 2$ and set $\dot{q} = v$

Let us calculate how the Hamiltonian evolves with Euler and Euler–Cromer

Euler

$$\begin{aligned} v_{n+1} = v_n - \varepsilon q_n, \qquad q_{n+1} &= q_n + \varepsilon v_n \\ H_{n+1} = v_{n+1}^2 + q_{n+1}^2 &= (v_n - \varepsilon q_n)^2 + (q_n + \varepsilon v_n)^2 \\ &= v_n^2 + \varepsilon^2 q_n^2 - 2\varepsilon v_n q_n + q_n^2 + \varepsilon^2 v_n^2 + 2\varepsilon v_n q_n \\ &= (1 + \varepsilon^2) H_n > H_n \end{aligned}$$

The energy is steadily increasing!

Harmonic oscillator

$$\ddot{q} + q = 0 \qquad H = \dot{q}^2 + q^2$$

Euler

$$v_{n+1} = v_n - \varepsilon q_n, \qquad q_{n+1} = q_n + \varepsilon v_n$$

$$H_{n+1} = (1 + \varepsilon^2) H_n > H_n$$

Euler–Cromer

$$v_{n+1} = v_n - \varepsilon q_n, \qquad q_{n+1} = q_n + \varepsilon v_{n+1} = (1 - \varepsilon^2) q_n + \varepsilon v_n$$

$$\begin{aligned} H_{n+1} &= v_{n+1}^2 + q_{n+1}^2 \\ &= H_n + \varepsilon^2 (v_n^2 - q_n^2) - 2\varepsilon^3 v_n q_n + \varepsilon^4 q_n^2 \end{aligned}$$

For the exact solution, the ε^2 and ε^3 term will average to 0 over a complete oscillation.

The midpoint method

Can we reduce the $\mathcal{O}(\varepsilon)$ error in the Euler method?
Try taking the symmetric derivative

Step 1

Simple equation $$\frac{dy}{dx} = f(x)$$

The symmetric derivative gives

$$\frac{y(x-\delta) - y(x+\delta)}{2\delta} = f(x) \quad \Longrightarrow \quad y(x+\delta) = y(x-\delta) + 2\delta f(x)$$

Take $\varepsilon = 2\delta,\ x_n = x - \delta,\ x_{n+1} = x + \delta,\ x = x_{\text{mid}}$

This gives y_{n+1} from y_n and f evaluated at the midpoint:

$$y_{n+1} = y_n + \varepsilon f(x_n + \varepsilon/2)$$

The midpoint method

Step 2

What if $f = f(x, y)$?

Take a ‘trial step’ to the midpoint:

$$y_{\text{mid}} = y_n + \frac{\varepsilon}{2} f(x_n) \qquad \text{[Looks like Euler!]}$$

Then use this to evaluate f at the midpoint:

$$y_{n+1} = y_n + \varepsilon f(x_{\text{mid}}, y_{\text{mid}})$$

Error analysis

1. Error in derivative

$$\Delta_x^C y = \frac{dy}{dx} + \mathcal{O}(\varepsilon^2)$$

$$\implies \quad \frac{y^{\text{true}}(x_{n+1}) - y^{\text{true}}(x_n)}{\varepsilon} = f(x_{\text{mid}}, y^{\text{true}}(x_{\text{mid}})) + \mathcal{O}(\varepsilon^2)$$

2. Error in y_{mid}

$$y_{\text{mid}} = y^{\text{true}}(x + \varepsilon/2) + \mathcal{O}(\varepsilon^2) \qquad \text{[the Euler error]}$$

3. Error in $f(x_{\text{mid}}, y_{\text{mid}})$

$$f(x_{\text{mid}}, y_{\text{mid}}) = f(x_{\text{mid}}, y^{\text{true}}(x_{\text{mid}}) + \mathcal{O}(\varepsilon^2)) = f^{\text{true}} + \mathcal{O}(\varepsilon^2)$$

4. Error per step: $\varepsilon\mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^3)$
5. Total error: $\mathcal{O}(\varepsilon^2)$

The midpoint method is the starting point for Runge–Kutta methods

Summary

- Instability in Euler method:
  - 1st order ODE: too large stepsize → runaway solutions
  - Hamiltonian systems: H increases with every step
  - Euler–Cromer cures instability in hamiltonian systems
- Midpoint method: gives improvement in accuracy